RéGA, 13 juin 2012

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Deligne's Riemann Roch Theorem

Some standing assumptions:

everything is over an alg. closed field $k=\bar{k}$

My varieties will (almost always) be smooth

Riemann-Roch for curves:

$C$ curve

$D = \sum_{P \in C} n_P P$ divisor on $C$

let $L = \mathcal{O}(D)$ be the associated line bundle.

Then, if $\chi(L) = h^0(L) - h^1(L)$

R-R: $\chi(L) = \deg L + 1 - g$, $g = g(C)$

Here: $\deg L = \sum n_P$

Sketch of proof

$\chi(F) = \sum (-1)^i h^i(\mathcal{F})$ has the property

that for an exact sequence of sheaves:

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \,/_C$

$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$

This means that an expression of the form

$$\chi(L) = \chi(L - \mathcal{O}_C) + \chi(\mathcal{O}_C)$$

where $\chi(L - \mathcal{O}_C) := \chi(L) - \chi(\mathcal{O}_C)$

Two statements hold:

$$\chi(L - \mathcal{O}_C) = \deg L$$

$$\chi(\mathcal{O}_C) = 1 - g_C$$

$$\| $$

$$\underset{\underset{1}{\|}}{h^0(\mathcal{O}_C)} - \underset{\underset{g_C}{\|}}{h^1(\mathcal{O}_C)} \quad \text{by Serre duality}$$

The first point can be proven by playing with exact sequences of the form

$$0 \to L(-p) \to L \to L|_p \to 0 \qquad \square$$

General interest is understanding of $h^0(L)$, but somehow we can mostly say something about $\chi(L)$.

Next, what does the Riemann-Roch thm say for surfaces?

Want to study $\chi(L)$ on a surface by rewriting $\chi(L) = \chi(L - C) + \chi(C)$

I need some intersection theory on a surface: $S$ proj. surface

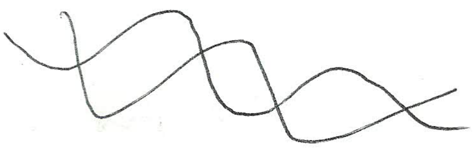
$$\mathrm{Div}(S) = \{ \sum n_D D \} \text{ divisors on } S$$

Int. theory is a certain

$$\mathrm{Div}(S) \times \mathrm{Div}(S) \to \mathbb{Z}$$

When $D, E$ irred. divisors, $D \neq E$

$$D \cdot E = \sum_{P \in S} (D, E)_P$$



In general, one sets

$$(D, E)_P = \dim_k \mathcal{O}_{S,P} / (f, g)$$

where $f, g$ are local equations for divisors

- Some properties

Symmetric: $D \cdot E = E \cdot D$

linear in both factors:

$$(D + D') \cdot E = D \cdot E + D' \cdot E$$

If $D \sim D'$ linear equivalent, then $D \cdot E = D' \cdot E$

This can be slightly refined.

I write $(D \cdot E) = \sum (D \cdot E)_P$ but this actually defines something finer:

a zero cycle on the surface

$Z_0(S) = \{ \sum n_P \cdot P \}$  $P$ are 0-dim (closed) points

$$(D \cdot E) = \sum (D \cdot E)_P \, P$$

$$\mathrm{Div}\, S \times \mathrm{Div}\, S \longrightarrow Z_0(S) \xrightarrow{\deg} \mathbb{Z}$$

This is only well defined when the divisors intersect properly (i.e. no common components)

$$\begin{tikzcd}
\mathrm{Div}\, S \times \mathrm{Div}\, S \arrow[rr] \arrow[dr] & & Z_0(S)/\text{rat. eq.} \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}$$

lin. eq.

<u>Def.</u> Given a variety $X$, two $i$-dimensional

cycles $\sum n_Z Z$, $\sum n_{Z'} Z'$, $\dim Z = i$ are rationally equivalent if there exists a closed subvariety

$$\begin{tikzcd} V \arrow[r, hook] \arrow[dr, "\text{dominant}", "f"'] & X \times \mathbb{P}^1 \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$

such that

$$A - B = f^{-1}(0) - f^{-1}(\infty)$$

dim $i$-cycles mod. rat. eq. $= CH_i(X)$

$$\begin{tikzcd} CH_1(S) \times CH_1(S) \arrow[r] \arrow[dr] & CH_0(S) \arrow[d, "\deg"] \\ & \mathbb{Z} \end{tikzcd}$$

<u>Fact</u>: since $S$ smooth

$$CH_1(S) = \mathrm{Pic}(S) = \{\text{line bundles on } S\}/\text{iso}$$

We thus obtain a certain map:

$$\begin{tikzcd} \mathrm{Pic}(S) \times \mathrm{Pic}(S) \arrow[r] \arrow[dr] & CH_0(S) \arrow[d, "\deg"] \\ & \mathbb{Z} \end{tikzcd}$$

Cohomological interpretation of the intersection product

$$(D\cdot E) = \sum (D\cdot E)_p$$

$$(D\cdot E)_p = \dim_k \mathcal{O}_{S,p}/(f,g) = h^0(\mathcal{O}_{D\cap E})$$

$D, E$ irred. red. divisors

Then: $0 \to \mathcal{O}_E(-D|_E) \to \mathcal{O}_E \to \mathcal{O}_{D\cap E} \to 0$

$$h^0(\mathcal{O}_{D\cap E}) = \chi(\mathcal{O}_{D\cap E})$$

$$\chi(\mathcal{O}_E) = \chi(\mathcal{O}_{D\cap E}) + \chi(\mathcal{O}_E(-D|_E))$$

$$0 \to \mathcal{O}_S(-E) \to \mathcal{O}_S \to \mathcal{O}_E \to 0$$

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_E) + \chi(\mathcal{O}_S(-E))$$

to obtain:

$$(D\cdot E) = h^0(\mathcal{O}_{D\cap E}) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}(-E)) - \chi(\mathcal{O}(-D)) + \chi(\mathcal{O}(-D-E))$$

get formula for $(D\cdot E)$ only depending on $\mathcal{O}(-E)$, $\mathcal{O}(-D)$.

In fact: one obtains

$$(\mathcal{L}\cdot M) = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^{-1}) - \chi(M^{-1}) + \chi((\mathcal{L}\otimes M)^{-1})$$

$$= \chi(\mathcal{O}_S) - \chi(L) - \chi(M) + \chi(L\otimes M)$$

It allows to compute:

$\chi(L - \mathcal{O}_S)$ in terms of intersect. theory

$$2(\chi(L) - \chi(\mathcal{O}_S)) = [\chi(\mathcal{O}_S) - \chi(L) - \chi(\mathcal{O}_S) + \chi(L)]$$

but: if $\omega_S$ = dualizing sheaf = $\det \Omega_S$,

then Serre duality states

$$h^i(L) = h^{2-i}(L^{-1}\otimes \omega)$$

$$\Rightarrow \chi(L) = \chi(L^{-1}\otimes \omega)$$

$$\chi(\mathcal{O}_S) - \chi(L) - \underset{\parallel}{\chi(L)} + \underset{\parallel}{\chi(\mathcal{O}_S)} =$$

$$\chi(\mathcal{O}_S) - \chi(L) - \chi(L^{-1}\otimes\omega) + \chi(\omega_S)$$

$$= L\cdot(\omega_S - L)$$

Accomplished:

$$\chi(L - \mathcal{O}_S) = \frac{L\cdot(L - \omega_S)}{2}$$

We have:

$$\chi(L) = \chi(L - \mathcal{O}_S) + \chi(\mathcal{O}_S)$$
$$= \frac{L\cdot(L-\omega_S)}{2} + \chi(\mathcal{O}_S)$$

One more term to consider:

$$\chi(\mathcal{O}_S) \underset{\substack{\text{Noether}\\ \text{formula}}}{=} \frac{\omega_S\cdot\omega_S + c_2(\Omega_S)}{12}$$

This in turn is a consequence of Grothendieck's version of the Riemann-Roch formula.

$X$ scheme

$K_0(X)$ = Grothendieck group of vector bundles

$= \{\text{formal sums of vector bundles}\}\big/\substack{\text{exact}\\ \text{sequences}}$

$[E] = [E'] + [E'']$ if

$$0 \to E' \to E \to E'' \to 0$$

I already defined Chow groups

$$CH_i(X) = \text{dim } i\text{-cycles modulo rat. equivalence}$$

Suppose $f : X \to Y$ proper morphism of smooth varieties

$$f_! : K_0(X) \to K_0(Y)$$

$$f_* : CH_i(X) \to CH_i(Y)$$

defined as follows:

given a vector bundle $E$ on $X$

$$f_! E = f_* E - R^1 f_* E + R^2 f_* E - \dots$$

given an $i$-dim cycle $Z$; suppose irred. reduced

$$f_* Z = \begin{cases} [k(Z) : k(f(Z))]\, f(Z) & \text{if finite} \\ 0 & \text{otherwise} \end{cases}$$

Natural transformation $K_0(X) \xrightarrow{ch} \bigoplus CH_i(X)_{\mathbb{Q}}$

$$ch(L) = \exp(c_1(L)) = \sum \frac{c_1(L)^k}{k!}$$

$$ch(E) = rk\, E + c_1(E) + \dots$$

where for a vector bundle $E$

$$c_1(E) = c_1(\Lambda^{max} E) = [D] \in CH_{\dim X - 1}(X)$$

where $\mathcal{O}(D) = \Lambda^{max} E$.

Finally
$$Td(\Omega_{X/Y}) = 1 - \frac{c_1(\Omega_{X/Y})}{2} + \frac{c_1(\Omega_{X/Y})^2 + c_2(\Omega_{X/Y})}{12} + \dots$$

$$\begin{array}{ccc}
K_0(X) & \xrightarrow{ch(\cdot)Td(\Omega_{X/Y})} & \bigoplus CH_\bullet(X)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
K_0(Y) & \xrightarrow{ch(\cdot)} & \bigoplus CH_\bullet(Y)_{\mathbb{Q}}
\end{array}$$

$X$ = surface, $Y = pt$

$$\chi(L) = ch(f_* \mathcal{O}_S - R^1 f_* \mathcal{O}_S + R^2 f_* \mathcal{O}_S)$$
$$= f_*\left(\frac{c_1(\Omega_{X/Y})^2 + c_2(\Omega_{X/Y})}{12}\right)$$

There are however other terms in the Riemann-Roch theorem. On one hand, over a point, one gets a formula for $\chi(L)$ in terms of int. theory. This corresponds to $ch(E) = \mathrm{rg}\, E + c_1(f_* L - R^1 f_* L + R^2 f_* L)$

Suppose we have a family of curves (smooth) $f : C \to B$, then GRR gives an expression for

$$c_1(f_* L - R^1 f_* L) = f_*\left( \frac{c_1(L)\, c_1(L \otimes \omega^{-1}_{C/B})}{2} + \frac{c_1(\omega_{C/B})^2}{12} \right)$$

an equality in $CH_{\dim B - 1}(B)_{\mathbb{Q}} = \mathrm{Pic}(B)_{\mathbb{Q}}$

Basic question: equality of certain line bundle classes can it be realized on the level of actual line bundles?

can it be refined to the level of actual line bundles?

$B = \operatorname{Spec} k$ GRR gives an equality in

$\operatorname{Pic}(B)_{\mathbb{Q}}$ = trivial

First observation:

LHS has a natural interpretation not only as a class of line bundles but as a natural line bundle:

$$c_1(f_* L - R^1 f_* L) = c_1(f_* L) - c_1(R^1 f_* L) = c_1(\Lambda^{max} f_* L) - c_1(\Lambda^{max} R^1 f_* L)$$

$$= c_1((\Lambda^{max} f_* L) \otimes (\Lambda^{max} R^1 f_* L)^{-1})$$

the line bundle $\Lambda^{max} f_* L \otimes \Lambda^{max} R^1 f_* L$

represents the LHS.

This bundle: $\underset{\lambda(L)}{\overset{\shortparallel}{\det R f_* L}} = \det f_* L \otimes (\det R^1 f_* L)^{-1}$

is called the determinant of the cohomology.

Basic question: is there a Riemann-Roch type formula for $\lambda(L)$

$L \mapsto \lambda(L)$ is functorial wrt short exact sequences $0 \to E' \to E \to E'' \to 0$ $/C$

$$\lambda(E) \simeq \lambda(E') \otimes \lambda(E'')$$

[analogy: $\chi(E) = \chi(E') + \chi(E'')$]

<u>Example</u>: consider an elliptic curve with Weierstrass equation

$$E: \ y^2 = x^3 + ax + b$$

$$\lambda(w_E) = \det H^0(w_E) \otimes \underbrace{\det H^1(w_E)^{-1}}_{\text{trivial bundle by Serre duality}}$$

$$= H^0(w_E)$$

$\eta$ = invariant differential $\dfrac{dy}{3x^2+a} \in H^0(w_E)$

$\Delta = -16(4a^3 + 27b^2)$

Statement: $\quad \Delta \eta^{\otimes 12} \in H^0(\omega_E)^{\otimes 12} = \lambda(\omega)^{\otimes 12}$

which is more over independent of Weierstrass equation

This is the LHS of Deligne's Riemann Roch:

GRR: in $\mathrm{Pic}(B)_{\mathbb{Q}}$

$f: C \to B$

$$[\lambda(L)] = f_*\left(\frac{c_1(L)\,c_1(L \otimes \omega^{-1})}{2} + \frac{c_1(\omega)^2}{12}\right)$$

Basic question: can $f_*(c_1(L)\,c_1(L \otimes \omega^{-1}))$ be represented as a line bundle?

Deligne's bundles

$f: B \to C$ smooth family of curves

$L, M / C$ line bundles

Then there is a functorial association of a line bundle $\langle L, M \rangle$ on $B$.

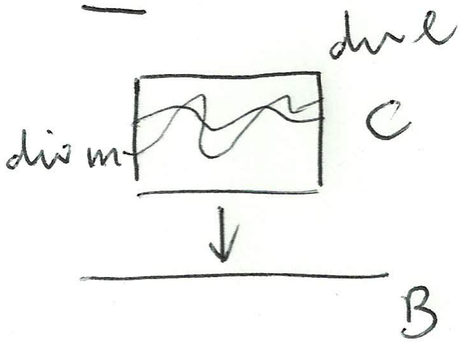
how to define such a bundle?

Suppose $B$ = point

$\langle L, M \rangle = \{$ vector space of symbols $\langle l, m \rangle$ where $l, m$ rational sections of $L$ & $M$ such that $\operatorname{div} l \cap \operatorname{div} m = \emptyset \}$


mod equiv. relation (Weil reciprocity)

Ex:




Then $\langle l, m \rangle$ is a trivialisation of $\langle L, M \rangle / B$ wherever $\operatorname{div} l|_{f^{-1}(b)} \cap \operatorname{div} m|_{f^{-1}(m)} = \emptyset$

family of curves $\langle l, m \rangle$

$\langle l, m \rangle$ section which might vanish when $\operatorname{div} l \cap \operatorname{div} m \neq \emptyset$

For $\mathbb{P}^1$ $\quad \underset{F}{\mathcal{O}(n)}, \underset{G}{\mathcal{O}(m)}$ $\quad \langle F, G \rangle \Leftrightarrow$ Theory of resultants

$$(\mathcal{L}.M) = \chi(\mathcal{O}_S) - \chi(\mathcal{L}) - \chi(M) + \chi(L \otimes M)$$

$$\langle L, M \rangle = \lambda(\mathcal{O}_S) \otimes \lambda(L)^{-1} \otimes \lambda(M)^{-1} \otimes \lambda(L \otimes M)$$

Same proof as before gives:

Serre duality

$$\lambda(L - \mathcal{O})^{\otimes 2} = \langle L, L \otimes \omega^{-1} \rangle$$

To compute

$$\lambda(L)^2 = \lambda(L - \mathcal{O})^{\otimes 2} \otimes \lambda(\mathcal{O})^{\otimes 2}$$
$$= \langle L, L \otimes \omega^{-1} \rangle \otimes \lambda(\mathcal{O})^{\otimes 2}$$

Mumford computed:

$$\lambda(\mathcal{O}_S)^{\otimes 12} = \langle \omega, \omega \rangle$$