The Tail Asymptotics of the Brownian Signature

Xi Geng
joint work with H. Boedihardjo

Department of Mathematical Sciences
Carnegie Mellon University

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The signature of a path

- Let $x : [0, T] \to \mathbb{R}^d$ be a continuous path with bounded variation.
- In 1954, Chen introduced the exponential homomorphism

$$S(x) = \sum_{n=0}^{\infty} \int_{0<t_1<\cdots<t_n<1} dx_{t_1} \otimes \cdots \otimes dx_{t_n}.$$

- Under the canonical basis $\{e_1, \cdots, e_d\}$,

$$S(x) = \sum_{n=1}^{\infty} \sum_{i_1,\cdots,i_n=1}^{d} \left(\int_{0<t_1<\cdots<t_n<T} dx_{t_1}^{i_1} \cdots dx_{t_n}^{i_n} \right) e_{i_1} \otimes \cdots \otimes e_{i_n}.$$

- $S(x)$ is known as the *signature* of the path x.
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Analytic and algebraic properties

Let $\mathbf{X}$ be a geometric p-rough path, and let $S(\mathbf{X})_{s,t} = (1, X^1_{s,t}, X^2_{s,t}, \cdots)$ be the signature of $\mathbf{X}$ over $[s,t]$.

Analytic property (Lyons 1998):

$$|X^n_{s,t}| \leqslant \frac{C^n \omega(s,t)^{\frac{n}{p}}}{(n/p)!}.$$

Algebraic properties (Chen 1954, 1958):

1. $S:$ Space of B.V. paths $\to T((\mathbb{R}^d))$ is a homomorphism.
2. $S(x)$ satisfies the shuffle product formula:

$$\mathrm{e}^*_I(S(x)) \cdot \mathrm{e}^*_J(S(x)) = \sum_{\sigma \in \mathrm{Shuffle}(|I|,|J|)} \mathrm{e}^*_{\sigma^{-1}(I \sqcup J)}(S(x)).$$

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The uniqueness result for signature

- Uniqueness result for signature (Hambly-Lyons 2010, Boedihardjo-G.-Lyons-Yang 2016): Let $\mathbf{X}$ be a geometric rough path. $\mathbf{X}$ has trivial signature if and only if it is tree-like, in the sense that it can be lifted to a continuous loop in some real tree.
- Every geometric rough path is uniquely determined by its signature up to tree-like equivalence.
- Every tree-like equivalence class contains a unique representative, called the *tree-reduced path*, which does not contain any tree-like pieces.
- Uniqueness result $\rightsquigarrow$ one-to-one correspondence between tree-reduced paths and their signatures.

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The length conjecture

- Question: can we recover intrinsic geometric quantities of a tree-reduced path from its signature/tail asymptotics of signature?

Conjecture (Length conjecture)

Let $x : [0,1] \to V$ be a continuous B.V. path over a finite dimensional normed vector space V, and let $g = (1, g_1, g_2, \cdots)$ be its signature. Then

$$\mathrm{Length}(x) = \lim_{n\to\infty} \left(n!\|g_n\|_{\mathrm{proj}}\right)^{\frac{1}{n}},$$

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Hambly-Lyons' result for C^1-paths

- $V = \mathbb{R}^d$ is equipped with the Euclidean norm.
- Hambly-Lyons 2010: If $x \in C^1$ when parametrized in unique speed such that the modulus of continuity $\omega_{\dot{x}}(\varepsilon) = o(\varepsilon^{3/4})$, then the length conjecture holds.
- The fundamental idea of proof: look at the hyperbolic development of the underlying path x.
- Let

$$\mathbb{H}^d = \left\{ x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d} x_i^2 - x_{d+1}^2 = -1,\ x_{d+1} > 0 \right\}$$

be the d-dimensional hyperbloid.

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- The isometry group G of $\mathbb{H}^d$ is the space of $(d+1)\times(d+1)$-matrices Γ such that $\Gamma^* J \Gamma = J$, where $J = \mathrm{diag}(1,\cdots,1,-1)$.
- The lie algebra $\mathfrak{g}$ of G is the space of $(d+1)\times(d+1)$-matrices of the form

$$A = \begin{pmatrix} A_0 & b \\ b^* & 0 \end{pmatrix},$$

where A_0 is a skew-symmetric $d \times d$-matrix and $b \in \mathbb{R}^d$.

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- In general, if γ is a path in the Lie algebra $\mathfrak{g}$, we can develop γ to a path Γ on the Lie group G in the way

$$\Gamma_{t+\delta t} \approx \Gamma_t \cdot \exp(\delta \gamma_t).$$

- $\delta \Gamma_t = \Gamma_t \cdot \delta \gamma_t$.
- The solution to the equation

$$\begin{cases} d\Gamma_t = \Gamma_t \cdot d\gamma_t, \\ \Gamma_0 = \mathrm{Id}, \end{cases}$$

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- Define a natural embedding $F : \mathbb{R}^d \to \mathfrak{g}$ by

$$F: \mathbb{R}^d \to \mathfrak{g}, \quad x \mapsto \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}.$$

- Define Γ_t to be the solution to the linear ODE

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- Define $X_t \triangleq \Gamma_t o$, where $o = (0, \cdots, 0, 1)^*$ is the base point of the hyperboloid $\mathbb{H}^d$.
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Hambly-Lyons' result for C^1-paths

- The hyperbolic length of X = the Euclidean length of x.
- The development of a line segment is a geodesic.
- If x is a piecewise linear path, then X is a piecewise geodesic with the same edge lengths and intersection angles as x.
- The hyperbolic distance $d(X_1, o)$ between endpoints of X is

$$\cosh d(o, X_1) = \sum_{n=0}^{\infty} \int_{0<t_1<\cdots<t_{2n}<1} \langle d\gamma_{t_1}, d\gamma_{t_2} \rangle \cdots \langle d\gamma_{t_{2n-1}}, d\gamma_{t_{2n}} \rangle.$$

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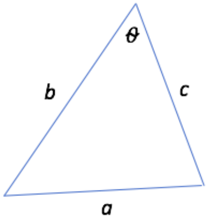
Let $\theta \in (0, \pi)$. For any hyperbolic triangle with edges a, b, c and with angle against a being θ, we have

$$b + c - a \leqslant \log \frac{2}{1 - \cos\theta}.$$



Hambly-Lyons' result for C^1-paths

- x: piecewise linear path with two edges and intersection angle θ. Define $L \triangleq \mathrm{Length}(x)$.
- For each $\lambda > 0$, let X^λ be the hyperbolic development of $\lambda \cdot x$.
- We have uniform estimate (in λ)

$$0 \leqslant \lambda L - d(X_1^\lambda, o) \leqslant \log \frac{2}{1 - \cos\theta}.$$

- $\lim_{\lambda\to\infty} d(X_1^\lambda, o)/\lambda = L$.

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$$\widetilde{L} \triangleq \sup_{n\geqslant 1} \left(n!\|g_n\|_{\mathrm{proj}}\right)^{\frac{1}{n}} \leqslant L.$$
- $d(X_1^\lambda, o) \leqslant \lambda \widetilde{L}$.
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Hambly-Lyons' result for C^1-paths

- $\cosh d(X_1^\lambda, o) = \sum_{n=0}^{\infty} \lambda^{2n} \int_{0<t_1<\cdots<t_{2n}<1} \langle d\gamma_{t_1}, d\gamma_{t_2}\rangle \cdots \langle d\gamma_{t_{2n-1}}, d\gamma_{t_{2n}}\rangle.$
- Define
$$\widetilde{L} \triangleq \sup_{n\geqslant 1} \left(n!\|g_n\|_{\mathrm{proj}}\right)^{\frac{1}{n}} \leqslant L.$$
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- The analysis for a general piecewise linear path is similar:

$$0 \leqslant \lambda L - d(X_1^\lambda, 0) \leqslant N \cdot \log \frac{2}{1 - \cos\theta},$$

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- In the rough path case, in view of the factorial estimate on signature, a natural renormalization of $\|g_n\|_{\mathrm{proj}}$ will be

$$\left(\left(\frac{n}{p}\right)!\|g_n\|_{\mathrm{proj}}\right)^{\frac{p}{n}}.$$

- We should look at "lim sup" instead of "lim" in the tail asymptotics (e.g. consider the 2 rough-path $\exp(t[\mathrm{e}_1,\mathrm{e}_2])$).
- If length conjecture for a B.V. path x is true, then for any $p>1$,

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The tail asymptotics of the Brownian signature

- Let $B_t = (B_t^1, \cdots, B_t^d)$ be a d-dimensional Brownian motion over $[0,1]$.
- B_t has finite quadratic variation in the mean sense.
- For $0 \leqslant s \leqslant t \leqslant 1$, define

$$\widetilde{L}_{s,t} \triangleq \limsup_{n\to\infty} \left(\left(\frac{n}{2}\right)! \left\| \int_{s<t_1<\cdots<t_n<t} \circ dB_{t_1} \otimes \cdots \otimes \circ dB_{t_n} \right\| \right)^{\frac{2}{n}}.$$

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Theorem (Boedihardjo-G. 2017)

Let $\mathbb{R}^d$ be equipped with the Euclidean norm. Then there exists a deterministic constant κ_d depending only on d, such that

$$\mathbb{P}\left(\widetilde{L}_{s,t} = \kappa_d (t-s) \quad \text{for all } 0 \leqslant s \leqslant t \leqslant 1\right) = 1.$$

Moreover, the constant κ_d satisfies

$$\frac{d-1}{2} \leqslant \kappa_d \leqslant d^2.$$

The tail asymptotics of the Brownian signature

Lemma

Define

$$\mathbb{B}_{s,t}^{n;i_1,\cdots,i_n} \triangleq \int_{s<t_1<\cdots<t_n<t} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_n}^{i_n}.$$

Then

$$\mathbb{E}\left[\left|\mathbb{B}_{s,t}^{n;i_1,\cdots,i_n}\right|\right] \leqslant \left(\frac{1}{2}+\sqrt{2}\right)\left(\frac{\mathrm{e}}{\sqrt{2\pi}}\right)^{\frac{1}{2}} \frac{2^{\frac{n}{2}}}{(n-2)^{\frac{1}{4}}\sqrt{n!}}(t-s)^{\frac{n}{2}}.$$

Main points of proof:

- By the shuffle product formula, the square of signature in degree n can be read off from the signature in degree $2n$.
- Second moment of $\mathbb{B}^n$ can be estimated by using the explicit formula for the expected signature of Brownian motion.

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Proposition

For each pair of $s<t$, we have

$$\mathbb{P}\left(\widetilde{L}_{s,t}\leqslant d^2(t-s)\right)=1.$$

Main point of proof:

- A Borel-Cantelli type argument $\rightsquigarrow$ for each $r>t-s$, with probability one,

$$\|\mathbb{B}_{s,t}^n\|_{\mathrm{proj}}\leqslant\frac{Cd^n2^{\frac{n}{2}}}{(n-2)^{\frac{1}{4}}\sqrt{n!}}r^{\frac{n}{2}}$$

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Let $\mathbf{X}$ *be a rough path and let* $p \geqslant 1$. *Define*

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Then $(s,t) \mapsto \widetilde{l}_{s,t}$ *is sub-additive, i.e.*

$$\widetilde{l}_{s,t} \leqslant \widetilde{l}_{s,u} + \widetilde{l}_{u,t}$$

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Main points of proof: manipulation of Chen's identity and the neo-classical inequality.

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- Weak law of large numbers $\implies \widetilde{L}_{s,t} \leqslant \mathbb{E}\left[\widetilde{L}_{s,t}\right]$.

The null set can be chosen to be independent of s,t: strengthen all the previous estimates to the corresponding maximal inequalities.

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- The more interesting part: a lower estimate on κ_d.
- Define
$$\widetilde{L}_t \triangleq \limsup_{n\to\infty}\left(\left(\frac{n}{2}\right)!\|\mathbb{B}^n_{0,t}\|_{\mathrm{proj}}\right)^{\frac{2}{n}}.$$
- For each $\lambda > 0$, define Γ^λ_t to be the unique solution to the Stratonovich type SDE
$$\begin{cases} d\Gamma^\lambda_t = \lambda\Gamma^\lambda_t F(\circ dB_t), & t\in[0,1], \\ \Gamma^\lambda_0 = \mathrm{Id}. \end{cases}$$
- The hyperbolic development of Brownian motion: $X^\lambda_t \triangleq \Gamma^\lambda_t o$.
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With probability one, we have

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Main points of proof:

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$$\mathbb{E}\left[(h_t^{\lambda})^{-\mu}\right]\leqslant \exp\left(-\frac{\lambda^2\mu(d-1-\mu)t}{2}\right).$$

Sketch of proof:

- Rewrite the development equation in Itô form, we get

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Corollary

For almost every ω, the path $t \mapsto \mathbf{B}_t\omega$ is tree-reduced. In particular, with probability one, the Brownian rough path is uniquely determined by its signature up to reparametrization.

Proof:

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- Recall that with probability one, the Brownian motion B_t has a canonical lifting $\mathbf{B}_t$ as geometric p-rough paths with $2 < p < 3$. $\mathbf{B_t}$ is known as the Brownian rough path.

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For almost every ω, the path $t \mapsto \mathbf{B}_t\omega$ is tree-reduced. In particular, with probability one, the Brownian rough path is uniquely determined by its signature up to reparametrization.

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There exists a $\mathbb{P}$-null set $\mathscr{N}$, such that for any two distinct $\omega_1, \omega_2 \notin \mathscr{N}$, $\mathbf{B}(\omega_1)$ and $\mathbf{B}(\omega_2)$ cannot be equal up to a reparametrization. In particular, with probability one, the Brownian rough path together with its natural parametrization is uniquely determined by its signature.

Proof:

- Pick the null set $\mathscr{N}$ as in the main result.
- Suppose that $\mathbf{B}_t(\omega_2) = \mathbf{B}_{\sigma(t)}(\omega_1)$.
- Then $\widetilde{L}_{0,t}(\omega_2) = \widetilde{L}_{0,\sigma(t)}(\omega_1)$.
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- Given the signature g of $\mathbf{B}(\omega)$, the uniqueness result implies that the image of the signature path of $\mathbf{B}(\omega)$ is uniquely determined by g.
- For each $\xi = (1, \xi_1, \xi_2, \cdots)$ on the image, define

$$\|\xi\| \triangleq \limsup_{n\to\infty} \left(\left(\frac{n}{2}\right)! \|\xi_n\|_{\mathrm{proj}} \right)^{\frac{2}{n}}.$$

- Then $\mathbf{B}_{\|\xi\|/\kappa_d}(\omega) = \pi^{(2)}(\xi)$.

Remark

The result is stronger than the uniqueness result proved by Le Jan and Qian .

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Further questions

- Question 1: What is the exact value of κ_d?
- Question 2: What is the meaning of κ_d? Does it correspond to some sort of quadratic variation of the Brownian rough path?
- Question 3: Is it true that with probability one, no two *sampe paths* of Brownian motion can be equal up to a reparametrization?
- Question 3': We know that with probability one, the lifting of piecewise linear interpolation of Brownian motion converges to the Brownian rough path. But the $\mathbb{P}$-null set depends on the choice of the piecewise linear approximation. Can we make the null set universal, so that any *arbitrary* piecewise linear approximation gives the same Brownian rough path?

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The end

Thank you very much for your attention!