On level sets in the Heisenberg group

Dario Trevisan *

*Joint work with V. Magnani (UNIPI) and E. Stepanov (S.Pb UNIV. & STEKLOV)
### Implicit function theorem

Regular level sets of a $C^1$ map between Euclidean spaces have a local $C^1$ parametrization.

### Problem

What happens if we replace Euclidean spaces with more general Lie groups?

We study the simplest non-trivial case: $F : H \approx \mathbb{R}^3 \to \mathbb{R}^k$. 
Implicit function theorem

Regular level sets of a $C^1$ map between Euclidean spaces have a local $C^1$ parametrization.

Problem

What happens if we replace Euclidean spaces with more general Lie groups?

We study the simplest non-trivial case: $F : H \approx \mathbb{R}^3 \rightarrow \mathbb{R}^k$. 
Structure of the talk

1. Introduction
2. Heisenberg group
3. LSDE: formulation
4. LSDE: well-posedness
5. Towards higher dimensional level sets
The group $\mathbb{H}$ is a non-commutative Lie group, with two generators.

- On $\mathbb{H} = \mathbb{R}^3$, $x = (x^1, x^2, x^3)$, consider the two (horizontal) vector fields
  \[
  X_1(x) := \partial_1 - x^2 \partial_3 \quad X_2(x) := \partial_2 + x^1 \partial_3
  \]
  \[
  [X_1, X_2] = [\partial_1 - x^2 \partial_3, \partial_2 + x^1 \partial_3] = 2\partial_3 \quad \text{(Hörmander condition)}.
  \]

- Dual description: contact 1-form
  \[
  \theta = dx^3 + x^2 dx^1 - x^1 dx^2 \quad \Rightarrow \quad d\theta = -2 dx^1 \wedge dx^2
  \]

- Horizontal tangent at $x \in \mathbb{H}$ is span \{\(X_1(x), X_2(x)\}\} = Ker \(\theta_x\).
The group $\mathbb{H}$ is a non-commutative Lie group, with two generators.

- On $\mathbb{H} = \mathbb{R}^3$, $x = (x^1, x^2, x^3)$, consider the two (horizontal) vector fields
  \[
  X_1(x) := \partial_1 - x^2 \partial_3 \quad X_2(x) := \partial_2 + x^1 \partial_3
  \]
  \[
  [X_1, X_2] = [\partial_1 - x^2 \partial_3, \partial_2 + x^1 \partial_3] = 2\partial_3 \quad \text{(Hörmander condition)}.
  \]

- Dual description: contact 1-form
  \[
  \theta = dx^3 + x^2 dx^1 - x^1 dx^2 \quad \Rightarrow \quad d\theta = -2 dx^1 \wedge dx^2
  \]

- Horizontal tangent at $x \in \mathbb{H}$ is span \(\{X_1(x), X_2(x)\} = \text{Ker} \theta_x\).
The group $\mathbb{H}$ is a non-commutative Lie group, with two generators.

- On $\mathbb{H} = \mathbb{R}^3$, $x = (x^1, x^2, x^3)$, consider the two (horizontal) vector fields
  \[ X_1(x) := \partial_1 - x^2 \partial_3 \quad X_2(x) := \partial_2 + x^1 \partial_3 \]

  \[ [X_1, X_2] = [\partial_1 - x^2 \partial_3, \partial_2 + x^1 \partial_3] = 2\partial_3 \quad \text{(Hörmander condition)}. \]

- Dual description: contact 1-form
  \[ \theta = dx^3 + x^2 dx^1 - x^1 dx^2 \quad \Rightarrow \quad d\theta = -2dx^1 \wedge dx^2 \]

- Horizontal tangent at $x \in \mathbb{H}$ is span $\{X_1(x), X_2(x)\} = \text{Ker } \theta_x$.  

The group $\mathbb{H}$ is a non-commutative Lie group, with two generators.

- On $\mathbb{H} = \mathbb{R}^3$, $x = (x^1, x^2, x^3)$, consider the two (horizontal) vector fields
  \begin{align*}
  X_1(x) &:= \partial_1 - x^2 \partial_3 \\
  X_2(x) &:= \partial_2 + x^1 \partial_3
  \end{align*}

  \[ [X_1, X_2] = [\partial_1 - x^2 \partial_3, \partial_2 + x^1 \partial_3] = 2 \partial_3 \quad \text{(Hörmander condition)}. \]

- Dual description: contact 1-form
  \[ \theta = dx^3 + x^2 dx^1 - x^1 dx^2 \quad \Rightarrow \quad d\theta = -2 dx^1 \wedge dx^2 \]

- **Horizontal tangent** at $x \in \mathbb{H}$ is span $\{X_1(x), X_2(x)\} = \text{Ker} \, \theta_x$. 
Heisenberg group: curves and distance

- A (smooth) curve $\eta : I \to \mathbb{H}$ is horizontal if, for $t \in I$,
  \[
  \theta_{\eta_t}(\dot{\eta}_t) = \dot{\eta}_t^3 + \eta_t^2 \dot{\eta}_t^1 - \eta_t^1 \dot{\eta}_t^2 = 0.
  \]

- Imposing $X_1(x) X_2(x)$ are orthonormal $\Rightarrow$ CC-distance
  \[
  d(x, y) := \inf \left\{ \int_0^1 |\dot{\eta}_t| : \eta \text{ horizontal}, \eta_0 = x, \eta_1 = y. \right\}
  \]

- Equivalence
  \[
  d(x, y) \approx |y^1 - x^1| + |y^2 - x^2| + |\vartheta_{xy}|^{1/2},
  \]
  where a “discrete” contact form appears
  \[
  \vartheta_{xy} := (y^3 - x^3) + x^2(y^1 - x^1) - x^1(y^2 - x^2).
  \]
  (Recall $\theta = dx^3 + x^2 dx^1 - x^1 dx^2$.)
Heisenberg group: curves and distance

- A (smooth) curve \( \eta : I \to \mathbb{H} \) is \textbf{horizontal} if, for \( t \in I \),

\[
\theta_{\eta_t}(\dot{\eta}_t) = \dot{\eta}_t^3 + \eta_t^2 \dot{\eta}_t^1 - \eta_t^1 \dot{\eta}_t^2 = 0.
\]

- Imposing \( X_1(x) X_2(x) \) are orthonormal \( \Rightarrow \) CC-distance

\[
d(x, y) := \inf \left\{ \int_0^1 |\dot{\eta}_t| : \eta \text{ horizontal, } \eta_0 = x, \eta_1 = y \right\}
\]

- Equivalence

\[
d(x, y) \approx |y^1 - x^1| + |y^2 - x^2| + |\vartheta_{xy}|^{1/2},
\]

where a “discrete” contact form appears

\[
\vartheta_{xy} := (y^3 - x^3) + x^2(y^1 - x^1) - x^1(y^2 - x^2).
\]

(Recall \( \theta = dx^3 + x^2 dy^1 - x^1 dy^2 \).
Examples

1. Horizontal curve $\leftrightarrow$

$$\eta_t^3 - \eta_s^3 = \int_s^t \eta_r^2 \dot{\eta}_r^1 \, dr - \int_r^t \eta_r^1 \dot{\eta}_r^2 \, dr.$$

If $(\eta_1^t, \eta_2^t) = (\eta_1^s, \eta_2^s) \rightarrow \eta_t^3 - \eta_s^3 = \text{signed area}.$

2. If $(\eta_1^t, \eta_2^t)$ are $\frac{1+\alpha}{2}$-Hölder continuous $\rightarrow$ horizontal lift

$$\theta_{\eta_t}(\dot{\eta}_t) = 0$$

in the sense Young integrals or in the “incremental” sense

$$\vartheta_{\eta_s\eta_t} = (\eta_t^3 - \eta_s^3) + \eta_s^2(\eta_1^t - \eta_1^s) - \eta_s^1(\eta_2^t - \eta_2^s) = o(t - s)$$

3. If $\alpha = 0$, pure area rough path

$$n^{-1/2}(\cos(nt), \sin(nt)) \quad n \rightarrow \infty.$$

The limit of horizontal lifts is not horizontal!
We “measure” regularity of $F : \mathbb{H} \to \mathbb{R}^k$ in terms of horizontal derivatives

$$\nabla_h F(x) := (X_1 F(x), X_2 F(x)).$$

$p \in \mathbb{H}$ is non degenerate for $F$ if $\nabla_h F(p)$ has maximum rank.

For $\alpha \in (0, 1)$, $F \in C^1_h, \alpha$ if $x \mapsto \nabla_h F(x)$ is (well-defined and) $\alpha$-Hölder continuous, (w.r.t. $d$). ($F \in C^1_h$ if just continuous).

Fact: There are $F \in C^1, \alpha$ nowhere (Euclidean) differentiable on a set of positive Lebesgue measure.

Problem

Locally parametrize $F^{-1}(F(p))$ for $F \in C^1_h$ for non degenerate $p$'s.
We “measure” regularity of $F : \mathbb{H} \to \mathbb{R}^k$ in terms of horizontal derivatives

$$\nabla_h F(x) := (X_1 F(x), X_2 F(x)).$$

$p \in \mathbb{H}$ is non degenerate for $F$ if $\nabla_h F(p)$ has maximum rank.

For $\alpha \in (0, 1)$, $F \in C^{1,\alpha}_h$ if $x \mapsto \nabla_h F(x)$ is (well-defined and) $\alpha$-Hölder continuous, (w.r.t. $d$). ($F \in C^1_h$ if just continuous).

Fact: There are $F \in C^{1,\alpha}_h$ nowhere (Euclidean) differentiable on a set of positive Lebesgue measure.

Problem

Locally parametrize $F^{-1}(F(p))$ for $F \in C^1_h$ for non degenerate $p$'s.
Heisenberg group: regular maps

- We “measure” regularity of $F : \mathbb{H} \to \mathbb{R}^k$ in terms of horizontal derivatives
  \[
  \nabla_h F(x) := (X_1 F(x), X_2 F(x)).
  \]
  
  $p \in \mathbb{H}$ is non degenerate for $F$ if $\nabla_h F(p)$ has maximum rank.

- For $\alpha \in (0, 1)$, $F \in C^{1,\alpha}_h$ if $x \mapsto \nabla_h F(x)$ is (well-defined and) $\alpha$-Hölder continuous, (w.r.t. $d$). ($F \in C^1_h$ if just continuous).

- **Fact:** There are $F \in C^{1,\alpha}_h$ nowhere (Euclidean) differentiable on a set of positive Lebesgue measure.

*Problem*

Locally parametrize $F^{-1}(F(p))$ for $F \in C^1_h$ for non degenerate $p$'s.
We “measure” regularity of $F : \mathbb{H} \to \mathbb{R}^k$ in terms of horizontal derivatives

$$\nabla_h F(x) := (X_1 F(x), X_2 F(x)).$$

$p \in \mathbb{H}$ is non degenerate for $F$ if $\nabla_h F(p)$ has maximum rank

For $\alpha \in (0, 1)$, $F \in C^{1, \alpha}_h$ if $x \mapsto \nabla_h F(x)$ is (well-defined and) $\alpha$-Hölder continuous, (w.r.t. $d$). ($F \in C^1_h$ if just continuous).

Fact: There are $F \in C^{1, \alpha}_h$ nowhere (Euclidean) differentiable on a set of positive Lebesgue measure.

Problem

Locally parametrize $F^{-1}(F(p))$ for $F \in C^1_h$ for non degenerate $p$’s.
Heisenberg group: regular maps

- We “measure” regularity of $F : \mathbb{H} \to \mathbb{R}^k$ in terms of horizontal derivatives
  \[
  \nabla_h F(x) := (X_1 F(x), X_2 F(x)).
  \]

  $p \in \mathbb{H}$ is non degenerate for $F$ if $\nabla_h F(p)$ has maximum rank

- For $\alpha \in (0, 1)$, $F \in C^{1,\alpha}_h$ if $x \mapsto \nabla_h F(x)$ is (well-defined and) $\alpha$-Hölder continuous, (w.r.t. $d$). ($F \in C^1_h$ if just continuous).

- Fact: There are $F \in C^{1,\alpha}_h$ nowhere (Euclidean) differentiable on a set of positive Lebesgue measure.

Problem

Locally parametrize $F^{-1}(F(p))$ for $F \in C^1_h$ for non degenerate $p$'s.
Literature on level sets in Heisenberg group, $F : \mathbb{H} \rightarrow \mathbb{R}^k$

$k = 1$: algebraic splitting phenomenon (Ambrosio-SerraCassano-Vittone, . . .) \Rightarrow “intrinsic graphs”, parametrized surfaces via group operation. (Interesting connection with non-linear PDE’s, recall talk by Katrin Fässler).


$k = 2$: Kozhevnikov (2011) \Rightarrow \beta$-Hölder continuous curves ($\beta < 1/2$) via a sub-Riemannian Reifenberg-type argument.

For $k = 2$, parametrizations are quite implicit: is a “good calculus” missing?

Main results (Magnani-Stepanov-T., 2016): $k = 2$.

- Explicit “Level Set Differential Equation” (LSDE).
- Prove existence, uniqueness, and stability w.r.t. approximations for $F \in C_h^{1,\alpha}$ ($\alpha > 0$) using tools from Young integration (Rough paths).
- Prove area formula and (re)-obtain a coarea formula for $F \in C_h^{1,\alpha}$. 
Literature on level sets in Heisenberg group, $F : \mathbb{H} \to \mathbb{R}^k$

$k = 1$: algebraic splitting phenomenon (Ambrosio-SerraCassano-Vittone, . . . )

$\Rightarrow$ “intrinsic graphs”, parametrized surfaces via group operation.

(Interesting connection with non-linear PDE’s, recall talk by Katrin Fäessler).


$k = 2$: Kozhevnikov (2011) $\Rightarrow$ $\beta$-Hölder continuous curves ($\beta < 1/2$) via a sub-Riemannian Reifenberg-type argument.

For $k = 2$, parametrizations are quite implicit: is a “good calculus” missing?

Main results (Magnani-Stepanov-T., 2016): $k = 2$.

- Explicit “Level-Set Differential Equation” (LSDE).
  - Prove existence, uniqueness, and stability w.r.t. approximations for $F \in C_{h,1,\alpha}^1 (\alpha > 0)$ using tools from Young integration (Rough paths).
  - Prove area formula and (re)-obtain a coarea formula for $F \in C_{h,1,\alpha}^1$. 

Literature on level sets in Heisenberg group, $F : \mathbb{H} \to \mathbb{R}^k$


$k = 2$: Kozhevnikov (2011) ⇒ $\beta$-Hölder continuous curves ($\beta < 1/2$) via a sub-Riemannian Reifenberg-type argument.

For $k = 2$, parametrizations are quite implicit: is a “good calculus” missing?

---

Main results (Magnani-Stepanov-T., 2016): $k = 2$.

- Explicit “Level Set Differential Equation” (LSDE).
- Prove existence, uniqueness, and stability w.r.t. approximations for $F \in C^{1,\alpha}_{h}$ ($\alpha > 0$) using tools from Young integration (Rough paths).
- Prove area formula and (re)-obtain a coarea formula for $F \in C^{1,\alpha}_{h}$. 

Literature on level sets in Heisenberg group, $F : \mathbb{H} \to \mathbb{R}^k$

$k = 1$: algebraic splitting phenomenon (Ambrosio-SerraCassano-Vittone, . . .) $
\Rightarrow$ “intrinsic graphs”, parametrized surfaces via group operation. 
(Interesting connection with non-linear PDE’s, recall talk by Katrin Fässler).


$k = 2$: Kozhevnikov (2011) $\Rightarrow$ $\beta$-Hölder continuous curves ($\beta < 1/2$) via a sub-Riemannian Reifenberg-type argument.

For $k = 2$, parametrizations are quite implicit: is a “good calculus” missing?

Main results (Magnani-Stepanov-T., 2016): $k = 2$.

- Explicit “Level Set Differential Equation” (LSDE).
- Prove existence, uniqueness, and stability w.r.t. approximations for $F \in C_h^{1,\alpha}$ ($\alpha > 0$) using tools from Young integration (Rough paths).
- Prove area formula and (re)-obtain a coarea formula for $F \in C_h^{1,\alpha}$. 
The Euclidean ODE argument

Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be $C^1$.

- Write $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, 2, 3$,
- $F = (F^1, F^2)$ and

$$\nabla F = \begin{pmatrix} \partial_1 F^1 & \partial_2 F^1 & \partial_3 F^1 \\
\partial_1 F^2 & \partial_2 F^2 & \partial_3 F^2 \end{pmatrix} = (\nabla_{12} F, \nabla_3 F) \text{ with } \nabla_{12} F \text{ invertible.}$$

Differentiating $F(\gamma_t) = c$,

$$\begin{pmatrix} \dot{\gamma}_t^1 \\
\dot{\gamma}_t^2 \\
\dot{\gamma}_t^3 \end{pmatrix} = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_t^3.$$

Impose $\dot{\gamma}_t^3 = 1$ and solve (Peano) for $(\gamma_t^1, \gamma_t^2)$. (Uniqueness of solutions?)
Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be $C^1$.

- Write $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, 2, 3$,
- $F = (F^1, F^2)$ and

$$\nabla F = \left( \begin{array}{ccc} \partial_1 F^1 & \partial_2 F^1 & \partial_3 F^1 \\ \partial_1 F^2 & \partial_2 F^2 & \partial_3 F^2 \end{array} \right) = (\nabla_{12} F, \nabla_3 F) \quad \text{with } \nabla_{12} F \text{ invertible.}$$

Differentiating $F(\gamma_t) = c$,

$$\left( \begin{array}{c} \dot{\gamma}_t^1 \\ \dot{\gamma}_t^2 \\ \dot{\gamma}_t^3 \end{array} \right) = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_t^3.$$

Impose $\dot{\gamma}_t^3 = 1$ and solve (Peano) for $(\gamma_t^1, \gamma_t^2)$. (Uniqueness of solutions?)
Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be $C^1$.

- Write $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, 2, 3$,
- $F = (F^1, F^2)$ and
  \[
  \nabla F = \begin{pmatrix}
    \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} & \frac{\partial F^1}{\partial x^3} \\
    \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} & \frac{\partial F^2}{\partial x^3}
  \end{pmatrix} = (\nabla_{12} F, \nabla_3 F) \quad \text{with } \nabla_{12} F \text{ invertible.}
  \]

Differentiating $F(\gamma_t) = c$,

\[
\begin{pmatrix}
  \dot{\gamma}_t^1 \\
  \dot{\gamma}_t^2 \\
  \dot{\gamma}_t^3
\end{pmatrix} = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_t^3.
\]

Impose $\dot{\gamma}_t^3 = 1$ and solve (Peano) for $(\gamma_t^1, \gamma_t^2)$. (Uniqueness of solutions?)
Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be $C^1$.

- Write $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, 2, 3$,
- $F = (F^1, F^2)$ and

$$\nabla F = \begin{pmatrix} \partial_1 F^1 & \partial_2 F^1 & \partial_3 F^1 \\ \partial_1 F^2 & \partial_2 F^2 & \partial_3 F^2 \end{pmatrix} = (\nabla_{12} F, \nabla_3 F)$$

with $\nabla_{12} F$ invertible.

Differentiating $F(\gamma_t) = c$,

$$\begin{pmatrix} \dot{\gamma}_t^1 \\ \dot{\gamma}_t^2 \\ \dot{\gamma}_t^3 \end{pmatrix} = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_t^3.$$

Impose $\dot{\gamma}_t^3 = 1$ and solve (Peano) for $(\gamma_t^1, \gamma_t^2)$. (Uniqueness of solutions?)
A naive approach to the LSDE

In $\mathbb{H} \sim \mathbb{R}^3$, recast the ODE

$$\begin{pmatrix} \dot{\gamma}_1^t \\ \dot{\gamma}_2^t \end{pmatrix} = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_3^t.$$

in terms of the horizontal derivatives $X_1 F, X_2 F$ (change of coordinates):

$$\begin{pmatrix} \dot{\gamma}_1^t \\ \dot{\gamma}_2^t \end{pmatrix} = - (\nabla_h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma} t (\dot{\gamma}_t).$$

In analogy with $\dot{\gamma}_3^t = 1$, set $\theta_{\gamma} t (\dot{\gamma}_t) = 1 \Rightarrow \text{non-horizontal, (vertical), curve.}$

Two difficulties:

1. The “vertical derivative” $\nabla_3 F$ may not be defined, even if $F \in C^{1,\alpha}$ with $0 < \alpha < 1$.

2. The intrinsic distance is $1/2$-Hölder along “vertical” directions $\Rightarrow \gamma$ is truly Hölder $\Rightarrow \dot{\gamma}_t$ is not defined.
A naive approach to the LSDE

In $\mathbb{H} \sim \mathbb{R}^3$, recast the ODE

\[
\begin{pmatrix}
\dot{\gamma}_t^1 \\
\dot{\gamma}_t^2 \\
\dot{\gamma}_t^3
\end{pmatrix} = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_t^3.
\]

in terms of the horizontal derivatives $X_1 F, X_2 F$ (change of coordinates):

\[
\begin{pmatrix}
\dot{\gamma}_t^1 \\
\dot{\gamma}_t^2 \\
\dot{\gamma}_t^3
\end{pmatrix} = - (\nabla_h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma_t}(\dot{\gamma}_t).
\]

In analogy with $\dot{\gamma}_t^3 = 1$, set $\theta_{\gamma}(\dot{\gamma}) = 1 \Rightarrow \text{non-horizontal, (vertical), curve.}$

Two difficulties:

1. The “vertical derivative” $\nabla_3 F$ may not be defined, even if $F \in C^{1,\alpha}$ with $0 < \alpha < 1$.
2. The intrinsic distance is $1/2$-Hölder along “vertical” directions $\Rightarrow \gamma$ is truly Hölder $\Rightarrow \dot{\gamma}_t$ is not defined.
In $\mathbb{H} \sim \mathbb{R}^3$, recast the ODE

$$\begin{pmatrix} \dot{\gamma}_t^1 \\ \dot{\gamma}_t^2 \end{pmatrix} = -\left(\nabla_{12} F(\gamma_t)\right)^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_t^3.$$

in terms of the horizontal derivatives $X_1 F, X_2 F$ (change of coordinates):

$$\begin{pmatrix} \dot{\gamma}_t^1 \\ \dot{\gamma}_t^2 \end{pmatrix} = -\left(\nabla_h F(\gamma_t)\right)^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma_t}(\dot{\gamma}_t).$$

In analogy with $\dot{\gamma}_t^3 = 1$, set $\theta_{\gamma_t}(\dot{\gamma}_t) = 1 \Rightarrow$ non-horizontal, (vertical), curve.

Two difficulties:

1. The “vertical derivative” $\nabla_3 F$ may not be defined, even if $F \in C^{1,\alpha}$ with $0 < \alpha < 1$.

2. The intrinsic distance is $1/2$-Hölder along “vertical” directions $\Rightarrow \gamma$ is truly Hölder $\Rightarrow \dot{\gamma}_t$ is not defined.
A naive approach to the LSDE

In $\mathbb{H} \sim \mathbb{R}^3$, recast the ODE

$$
\begin{pmatrix}
\dot{\gamma}_1^t \\
\dot{\gamma}_2^t
\end{pmatrix} = -\left(\nabla_{12} F(\gamma^t)\right)^{-1} \nabla_3 F(\gamma^t) \dot{\gamma}_3^t.
$$

in terms of the horizontal derivatives $X_1 F$, $X_2 F$ (change of coordinates):

$$
\begin{pmatrix}
\dot{\gamma}_1^t \\
\dot{\gamma}_2^t
\end{pmatrix} = -\left(\nabla_h F(\gamma^t)\right)^{-1} \nabla_3 F(\gamma^t) \theta_{\gamma^t}(\dot{\gamma}^t).
$$

In analogy with $\dot{\gamma}_3^t = 1$, set $\theta_{\gamma^t}(\dot{\gamma}^t) = 1 \Rightarrow$ non-horizontal, (vertical), curve.

Two difficulties:

1. The “vertical derivative” $\nabla_3 F$ may not be defined, even if $F \in C^{1,\alpha}$ with $0 < \alpha < 1$.

2. The intrinsic distance is $1/2$-Hölder along “vertical” directions $\Rightarrow \gamma$ is truly Hölder $\Rightarrow \dot{\gamma}_t$ is not defined.
In $\mathbb{H} \sim \mathbb{R}^3$, recast the ODE

$$
\begin{pmatrix}
\dot{\gamma}_t^1 \\
\dot{\gamma}_t^2
\end{pmatrix} = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_t^3.
$$

in terms of the horizontal derivatives $X_1 F, X_2 F$ (change of coordinates):

$$
\begin{pmatrix}
\dot{\gamma}_t^1 \\
\dot{\gamma}_t^2
\end{pmatrix} = - (\nabla_h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma_t}(\dot{\gamma}_t).
$$

In analogy with $\dot{\gamma}_t^3 = 1$, set $\theta_{\gamma_t}(\dot{\gamma}_t) = 1 \Rightarrow$ non-horizontal, (vertical), curve.

Two difficulties:

1. The “vertical derivative” $\nabla_3 F$ may not be defined, even if $F \in C^{1,\alpha}$ with $0 < \alpha < 1$.

2. The intrinsic distance is $1/2$-Hölder along “vertical” directions $\Rightarrow \gamma$ is truly Hölder $\Rightarrow \dot{\gamma}_t$ is not defined.
A naive approach to the LSDE

In $\mathbb{H} \sim \mathbb{R}^3$, recast the ODE

$$\begin{pmatrix} \dot{\gamma}_1^t \\ \dot{\gamma}_2^t \\ \dot{\gamma}_3^t \end{pmatrix} = - (\nabla_{12} F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \dot{\gamma}_3^t.$$  

in terms of the horizontal derivatives $X_1 F, X_2 F$ (change of coordinates):

$$\begin{pmatrix} \dot{\gamma}_1^t \\ \dot{\gamma}_2^t \\ \dot{\gamma}_3^t \end{pmatrix} = - (\nabla_h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma_t}(\dot{\gamma}_t).$$

In analogy with $\dot{\gamma}_3^t = 1$, set $\theta_{\gamma_t}(\dot{\gamma}_t) = 1 \Rightarrow$ non-horizontal, (vertical), curve.

Two difficulties:

1. The “vertical derivative” $\nabla_3 F$ may not be defined, even if $F \in C^{1,\alpha}$ with $0 < \alpha < 1$.

2. The intrinsic distance is $1/2$-Hölder along “vertical” directions $\Rightarrow \gamma$ is truly Hölder $\Rightarrow \dot{\gamma}_t$ is not defined.
Here’s a “rule of thumb” to define

\[
\begin{pmatrix}
\dot{\gamma}_1^t \\
\dot{\gamma}_2^t
\end{pmatrix} = - (\nabla_h F(\gamma^t))^{-1} \nabla_3 F(\gamma^t) \theta_{\gamma^t}(\dot{\gamma}^t), \quad \theta_{\gamma^t}(\dot{\gamma}^t) = 1.
\]

1. “Integrating” \( \theta_{\gamma^t}(\dot{\gamma}^t) = 1 \) gives \( \vartheta_{\gamma^t} = t - s \),

\[
\vartheta_{\vartheta_{\gamma^s}} = t - s, \quad d(\vartheta_{\vartheta_{\gamma^s}}, \gamma^t) \approx |t - s|^{1/2} \Rightarrow \gamma \text{ is (intrinsically) } 1/2\text{-Hölder.}
\]

2. Writing \( \partial_3 F = [X_1, X_2] F = X_1(X_2 F) - X_2(X_1 F) \) gives

\( \partial_3 F \) is \( (\alpha - 1) \)-Hölder.

3. The composition (\( ! \)) \( \partial_3 F(\gamma^t) \) is then \( \frac{1}{2} \cdot (\alpha - 1) \)-Hölder.

4. Integration w.r.t. \( t \) increases regularity of “one degree” \( \Rightarrow \)

\[
(\gamma^1, \gamma^2) \text{ is } \left[ \frac{1}{2} \cdot (\alpha - 1) + 1 \right] = \frac{1 + \alpha}{2} \text{-Hölder.}
\]

5. \( \frac{1 + \alpha}{2} \)-Hölder continuity is consistent with assumption 1, closing the circle.
Here’s a “rule of thumb” to define

$$
\begin{pmatrix}
\dot{\gamma}_1^t \\
\dot{\gamma}_2^t
\end{pmatrix} = - (\nabla_h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma t}(\dot{\gamma}_t), \quad \theta_{\gamma t}(\dot{\gamma}_t) = 1.
$$

1. “Integrating” \( \theta_{\gamma t}(\dot{\gamma}_t) = 1 \) gives \( \theta_{\gamma s \gamma t} = t - s \),

\[
d(\gamma_s, \gamma_t) \approx |t - s|^{1/2} \Rightarrow \gamma \text{ is (intrinsically) } 1/2\text{-Hölder.}
\]

2. Writing \( \partial_3 F = [X_1, X_2]F = X_1(X_2 F) - X_2(X_1 F) \) gives

\[
\partial_3 F \text{ is } (\alpha - 1)\text{-Hölder.}
\]

3. The composition (\(!\)) \( \partial_3 F(\gamma_t) \) is then \( \frac{1}{2} \cdot (\alpha - 1)\)-Hölder.

4. Integration w.r.t. \( t \) increases regularity of “one degree” ⇒

\[
\left(\gamma_1^1, \gamma_2^1 \right) \text{ is } \left[\frac{1}{2} \cdot (\alpha - 1) + 1 \right] = \frac{1 + \alpha}{2}\text{-Hölder.}
\]

5. \( \frac{1 + \alpha}{2} \)-Hölder continuity is consistent with assumption 1, closing the circle.
Here’s a “rule of thumb” to define

\[
\begin{pmatrix}
\dot{\gamma}_1^t \\
\dot{\gamma}_2^t
\end{pmatrix} = - (\nabla h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma_t}(\dot{\gamma}_t), \quad \theta_{\gamma_t}(\dot{\gamma}_t) = 1.
\]

1. “Integrating” \( \theta_{\gamma_t}(\dot{\gamma}_t) = 1 \) gives \( \vartheta_{\gamma_s \gamma_t} = t - s \),

\[
d(\gamma_s, \gamma_t) \approx |t - s|^{1/2} \Rightarrow \gamma \text{ is (intrinsically) } 1/2\text{-Hölder.}
\]

2. Writing \( \partial_3 F = [X_1, X_2] F = X_1 (X_2 F) - X_2 (X_1 F) \) gives

\[ \partial_3 F \text{ is } (\alpha - 1)\text{-Hölder.} \]

3. The composition (!) \( \partial_3 F(\gamma_t) \) is then \( \frac{1}{2} \cdot (\alpha - 1)\text{-Hölder.} \)

4. Integration w.r.t. \( t \) increases regularity of “one degree” \( \Rightarrow \)

\[
\left( \gamma^1, \gamma^2 \right) \text{ is } \left[ \frac{1}{2} \cdot (\alpha - 1) + 1 \right] = \frac{1 + \alpha}{2} \text{-Hölder.}
\]

5. \( \frac{1 + \alpha}{2} \text{-Hölder continuity is consistent with assumption 1, closing the circle.} \)
Here’s a “rule of thumb” to define

\[
\begin{pmatrix}
\dot{\gamma}_t^1 \\
\dot{\gamma}_t^2
\end{pmatrix} = - (\nabla h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_{\gamma_t}(\dot{\gamma}_t), \quad \theta_{\gamma_t}(\dot{\gamma}_t) = 1.
\]

1. “Integrating” \( \theta_{\gamma_t}(\dot{\gamma}_t) = 1 \) gives \( \vartheta_{\gamma_s \gamma_t} = t - s \),

\[d(\gamma_s, \gamma_t) \approx |t - s|^{1/2} \Rightarrow \gamma \text{ is (intrinsically) } 1/2\text{-Hölder.}\]

2. Writing \( \partial_3 F = [X_1, X_2] F = X_1(X_2 F) - X_2(X_1 F) \) gives

\( \partial_3 F \) is \((\alpha - 1)\text{-Hölder.}\)

3. The composition (!) \( \partial_3 F(\gamma_t) \) is then \( \frac{1}{2} \cdot (\alpha - 1)\text{-Hölder.}\)

4. Integration w.r.t. \( t \) increases regularity of “one degree” \( \Rightarrow \)

\[\left( \gamma^1, \gamma^2 \right) \text{ is } \left[ \frac{1}{2} \cdot (\alpha - 1) + 1 \right] = \frac{1 + \alpha}{2}\text{-Hölder.}\]

5. \( \frac{1 + \alpha}{2}\text{-Hölder continuity is consistent with assumption 1, closing the circle.} \)
Here’s a “rule of thumb” to define

\[
\begin{pmatrix}
\dot{\gamma}_t^1 \\
\dot{\gamma}_t^2
\end{pmatrix} = - (\nabla h F(\gamma^t))^{-1} \nabla_3 F(\gamma^t) \theta_{\gamma^t}(\dot{\gamma}_t), \quad \theta_{\gamma^t}(\dot{\gamma}_t) = 1.
\]

1. “Integrating” \( \theta_{\gamma^t}(\dot{\gamma}_t) = 1 \) gives \( \theta_{\gamma^s \gamma^t} = t - s \),

\[
d(\gamma^s, \gamma^t) \approx |t - s|^{1/2} \Rightarrow \gamma \text{ is (intrinsically) } 1/2\text{-Hölder.}
\]

2. Writing \( \partial_3 F = [X_1, X_2] F = X_1 (X_2 F) - X_2 (X_1 F) \) gives

\[
\partial_3 F \text{ is } (\alpha - 1)\text{-Hölder.}
\]

3. The composition (!) \( \partial_3 F(\gamma^t) \) is then \( \frac{1}{2} \cdot (\alpha - 1)\text{-Hölder.} \)

4. Integration w.r.t. \( t \) increases regularity of “one degree” \( \Rightarrow \)

\[
\left( \gamma^1, \gamma^2 \right) \text{ is } \left[ \frac{1}{2} \cdot (\alpha - 1) + 1 \right] = \frac{1 + \alpha}{2}\text{-Hölder.}
\]

5. \( \frac{1 + \alpha}{2} \)-Hölder continuity is consistent with assumption 1, closing the circle.
Here's a “rule of thumb” to define
\[
\begin{pmatrix}
\dot{\gamma}_1^t \\
\dot{\gamma}_2^t \\
\end{pmatrix} = - (\nabla h F(\gamma^t))^{-1} \nabla_3 F(\gamma^t) \theta_{\gamma^t}(\dot{\gamma}^t), \quad \theta_{\gamma^t}(\dot{\gamma}^t) = 1.
\]

1. “Integrating” \( \theta_{\gamma^t}(\dot{\gamma}^t) = 1 \) gives \( \vartheta_{\gamma s^t} = t - s \),
\[
d(\gamma_s, \gamma^t) \approx |t - s|^{1/2} \Rightarrow \gamma \text{ is (intrinsically) } 1/2\text{-Hölder.}
\]

2. Writing \( \partial_3 F = [X_1, X_2] F = X_1(X_2 F) - X_2(X_1 F) \) gives \( \partial_3 F \) is \((\alpha - 1)\)-Hölder.

3. The composition (!) \( \partial_3 F(\gamma^t) \) is then \( \frac{1}{2} \cdot (\alpha - 1)\)-Hölder.

4. Integration w.r.t. \( t \) increases regularity of “one degree” \( \Rightarrow \)
\[
(\gamma^1, \gamma^2) \text{ is } \left[ \frac{1}{2} \cdot (\alpha - 1) + 1 \right] = \frac{1 + \alpha}{2}\text{-Hölder.}
\]

5. \( \frac{1 + \alpha}{2} \)-Hölder continuity is consistent with assumption 1, closing the circle.
Heuristics

Here’s a “rule of thumb” to define

\[
\begin{pmatrix}
\dot{\gamma}_1^t \\
\dot{\gamma}_2^t
\end{pmatrix} = - (\nabla h F(\gamma_t))^{-1} \nabla_3 F(\gamma_t) \theta_t (\dot{\gamma}_t) , \quad \theta_t (\dot{\gamma}_t) = 1.
\]

1 “Integrating” \( \theta_t (\dot{\gamma}_t) = 1 \) gives \( \theta_s \gamma_t = t - s \),

\[
d(\gamma_s, \gamma_t) \approx |t - s|^{1/2} \Rightarrow \gamma \text{ is (intrinsically) } 1/2\text{-Hölder.}
\]

2 Writing \( \partial_3 F = [X_1, X_2]F = X_1(X_2 F) - X_2(X_1 F) \) gives

\[
\partial_3 F \text{ is } (\alpha - 1)\text{-Hölder.}
\]

3 The composition (!) \( \partial_3 F(\gamma_t) \) is then \( \frac{1}{2} \cdot (\alpha - 1)\text{-Hölder.} \)

4 Integration w.r.t. \( t \) increases regularity of “one degree” \( \Rightarrow \)

\[
(\gamma^1, \gamma^2) \text{ is } \left[ \frac{1}{2} \cdot (\alpha - 1) + 1 \right] = \frac{1 + \alpha}{2}\text{-Hölder.}
\]

5 \( \frac{1 + \alpha}{2}\text{-Hölder continuity is consistent with assumption 1, closing the circle.} \)
We adopt the point of view of “local descriptions” by finite increments and use physicists notation

\[ \delta \gamma^i_{st} = \gamma^i_t - \gamma^i_s, \quad \text{for } s, t \in I, \ i \in \{1, 2, 3\}. \]

The equation

\[ \theta_{\gamma_t}(\gamma_t) = \dot{\gamma}^3 + \gamma^2_t \dot{\gamma}^1_t - \gamma^1_t \dot{\gamma}^2_t = 1 \]

becomes our vertical equation

\[ \vartheta_{\gamma_s\gamma_t} = \delta \gamma^3_{st} + \gamma^2_s \delta \gamma^1_{st} - \gamma^2_s \delta \gamma^2_{st} = t - s + o(t - s). \]

(Compare with horizontal lift \( \vartheta_{\gamma_s\gamma_t} = o(t - s) \).)
We adopt the point of view of “local descriptions” by finite increments and use physicists notation

\[ \delta \gamma_{st}^i = \gamma_t^i - \gamma_s^i, \quad \text{for } s, t \in I, i \in \{1, 2, 3\}. \]

The equation

\[ \theta_{\gamma_t}(\dot{\gamma}_t) = \dot{\gamma}_t^3 + \gamma_t^2 \dot{\gamma}_t^1 - \gamma_t^1 \dot{\gamma}_t^2 = 1 \]

becomes our vertical equation

\[ \vartheta_{\gamma_s \gamma_t} = \delta \gamma_{st}^3 + \gamma_s^2 \delta \gamma_{st}^1 - \gamma_s^2 \delta \gamma_{st}^2 = t - s + o(t - s). \]

(Compare with horizontal lift \( \vartheta_{\gamma_s \gamma_t} = o(t - s). \))
We adopt the point of view of "local descriptions" by finite increments and use physicists notation

\[ \delta \gamma^i_{st} = \gamma^i_t - \gamma^i_s, \quad \text{for } s, t \in I, \ i \in \{1, 2, 3\}. \]

The equation

\[ \theta_{\gamma_t}(\dot{\gamma}_t) = \dot{\gamma}^3 + \gamma^2_t \dot{\gamma}^1_t - \gamma^1_t \dot{\gamma}^2_t = 1 \]

becomes our vertical equation

\[ \theta_{\gamma_s\gamma_t} = \delta \gamma^3 + \gamma^2_s \delta \gamma^1_t - \gamma^2_s \delta \gamma^2_t = t - s + o(t - s). \]

(Compare with horizontal lift \( \theta_{\gamma_s\gamma_t} = o(t - s). \))
The “horizontal” equations

Instead of “differentiating”, we use finite differences ⇒ horizontal Taylor expansion:

\[ F(y) - F(x) - \nabla_h F(x) \left( \frac{y^1 - x^1}{y^2 - x^2} \right) - \nabla_3 F(x) \vartheta_{xy} = R_{xy}. \]

Imposing \( F(\gamma_s) = F(\gamma_t) \) gives

\[ \left( \delta\gamma^1_{st}, \delta\gamma^2_{st} \right) = - (\nabla_h F(\gamma_s))^{-1} R_{\gamma_s \gamma_t} + o(t - s). \]

To avoid multiplication, a better formulation is

\[ \left( \delta\gamma^1_{st}, \delta\gamma^2_{st} \right) = - (\nabla_h F(p))^{-1} (R_{p \gamma_t} - R_{p \gamma_s}). \]
Instead of “differentiating”, we use finite differences $\Rightarrow$ horizontal Taylor expansion:

$$F(y) - F(x) - \nabla_h F(x) \left( \begin{array}{c} y^1 - x^1 \\ y^2 - x^2 \end{array} \right) - \nabla_3 F(x) \vartheta_{xy} = R_{xy}.$$ 

Imposing $F(\gamma_s) = F(\gamma_t)$ gives

$$\left( \delta \gamma^1_{st}, \delta \gamma^2_{st} \right) = -\left( \nabla_h F(\gamma_s) \right)^{-1} R_{\gamma_s \gamma_t} + o(t - s).$$

To avoid multiplication, a better formulation is

$$\left( \delta \gamma^1_{st}, \delta \gamma^2_{st} \right) = -\left( \nabla_h F(p) \right)^{-1} (R_{p\gamma_t} - R_{p\gamma_s}).$$
Instead of “differentiating”, we use finite differences ⇒ horizontal Taylor expansion:

\[
F(y) - F(x) - \nabla_h F(x) \left( \begin{array}{c} y^1 - x^1 \\ y^2 - x^2 \end{array} \right) - \nabla_3 F(x) \vartheta_{xy} = R_{xy}.
\]

Imposing \( F(\gamma_s) = F(\gamma_t) \) gives

\[
\left( \delta \gamma^1_{st}, \delta \gamma^2_{st} \right) = - (\nabla_h F(\gamma_s))^{-1} R_{\gamma_s \gamma_t} + o(t - s).
\]

To avoid multiplication, a better formulation is

\[
\left( \delta \gamma^1_{st}, \delta \gamma^2_{st} \right) = - (\nabla_h F(p))^{-1} (R_{p \gamma_t} - R_{p \gamma_s}).
\]
The LSDE

**Definition (LSDE)**

Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h$. We say $\gamma : I \to \mathbb{H}$ is a solution to the level set differential equation (LSDE) if it is continuous and

$$
\left\{ \begin{array}{l}
(\delta \gamma^1_{st}, \delta \gamma^2_{st}) = - (\nabla_h F(p))^{-1} (R_{p\gamma_t} - R_{p\gamma_s}) \\
\delta^2 \gamma_{st} = t - s + o(t - s)
\end{array} \right.
$$

for every $s, t \in I$.

- The “horizontal equation” yields that $t \mapsto F(\gamma_t)$ is constant.
- The “vertical equation” gives that

$$
d(\gamma_s, \gamma_t) \geq c |t - s|^{1/2},
$$

for $s, t$ sufficiently close.
The LSDE

**Definition (LSDE)**

Let \( p \) be non degenerate for \( F : \mathbb{H} \to \mathbb{R}^2, F \in C^1_h \). We say \( \gamma : I \to \mathbb{H} \) is a solution to the level set differential equation (LSDE) if it is continuous and

\[
\begin{align*}
(\delta \gamma^1_{st}, \delta \gamma^2_{st}) &= - (\nabla_h F(p))^{-1} (R_{p\gamma_t} - R_{p\gamma_s}) \\
\gamma_t^s &- \gamma_s^t = t - s + o(t - s)
\end{align*}
\]

for every \( s, t \in I \).

- The “horizontal equation” yields that \( t \mapsto F(\gamma_t) \) is constant.
- The “vertical equation” gives that

\[
d(\gamma_s, \gamma_t) \geq c |t - s|^{1/2},
\]

for \( s, t \) sufficiently close.
The LSDE

**Definition (LSDE)**

Let \( p \) be non degenerate for \( F : \mathbb{H} \to \mathbb{R}^2, F \in C^1_h \). We say \( \gamma : I \to \mathbb{H} \) is a solution to the level set differential equation (LSDE) if it is continuous and

\[
\begin{cases}
(\delta \gamma_{st}^1, \delta \gamma_{st}^2) &= - (\nabla_h F(p))^{-1} (R_{p\gamma_t} - R_{p\gamma_s}) \\
\partial_{\gamma_s \gamma_t} &= t - s + o(t - s)
\end{cases}
\]

for every \( s, t \in I \).

- The “horizontal equation” yields that \( t \mapsto F(\gamma_t) \) is constant.
- The “vertical equation” gives that

\[
d(\gamma_s, \gamma_t) \geq c |t - s|^{1/2},
\]

for \( s, t \) sufficiently close.
**Definition (LSDE)**

Let \( p \) be non degenerate for \( F : \mathbb{H} \to \mathbb{R}^2, F \in C^1_h \). We say \( \gamma : I \to \mathbb{H} \) is a solution to the level set differential equation (LSDE) if it is continuous and

\[
\left\{
\begin{align*}
(\delta \gamma^1_{st}, \delta \gamma^2_{st}) & = - (\nabla_h F(p))^{-1} (R_{p\gamma_t} - R_{p\gamma_s}) \\
\gamma_{s\gamma_t} & = t - s + o(t - s)
\end{align*}
\]

for every \( s, t \in I \).

- The “horizontal equation” yields that \( t \mapsto F(\gamma_t) \) is constant.
- The “vertical equation” gives that

\[
d(\gamma_s, \gamma_t) \geq c |t - s|^{1/2},
\]

for \( s, t \) sufficiently close.
The LSDE

Definition (LSDE)

Let \( p \) be non degenerate for \( F : \mathbb{H} \to \mathbb{R}^2 \), \( F \in C^1_h \). We say \( \gamma : I \to \mathbb{H} \) is a solution to the level set differential equation (LSDE) if it is continuous and

\[
\begin{aligned}
\left( \delta \gamma_1^1, \delta \gamma_2^2 \right) &= - (\nabla_h F(p))^{-1} (R_{p \gamma_t} - R_{p \gamma_s}) \\
\vartheta \gamma_s \gamma_t &= t - s + o(t - s)
\end{aligned}
\]

for every \( s, t \in I \).

- The “horizontal equation” yields that \( t \mapsto F(\gamma_t) \) is constant.
- The “vertical equation” gives that

\[
d(\gamma_s, \gamma_t) \geq c |t - s|^{1/2},
\]

for \( s, t \) sufficiently close.
Existence of solutions

**Theorem (Existence)**

Let $\alpha > 0$ and $p$ be non degenerate for $F : \mathbb{H} \rightarrow \mathbb{R}^2$, $F \in C^{1,\alpha}_h$. Then, there exists $\delta > 0$ and $\gamma : [-\delta, \delta] \rightarrow \mathbb{H}$ solving the LSDE with $\gamma_0 = p$.

- Proof via Leray-Schauder fixed point on a subset of $C^{1+\alpha/2}([-\delta, \delta]; \mathbb{R}^3)$.
- Need of $\alpha > 0$: use Young integral (sewing lemma) to move from

$$v_{\gamma_s\gamma_t} = t - s + o(t - s)$$

to

$$\delta \gamma^3_{st} = - \int_s^t \gamma^2_r \, d\gamma^1_r + \int_s^t \gamma^1_r \, d\gamma^2_r + (t - s).$$
Existence of solutions

Theorem (Existence)
Let $\alpha > 0$ and $p$ be non degenerate for $F : H \to \mathbb{R}^2$, $F \in C_{h,\alpha}^1$. Then, there exists $\delta > 0$ and $\gamma : [-\delta, \delta] \to H$ solving the LSDE with $\gamma_0 = p$.

- Proof via Leray-Schauder fixed point on a subset of $C^{\frac{1+\alpha}{2}}([-\delta, \delta]; \mathbb{R}^3)$.
- Need of $\alpha > 0$: use Young integral (sewing lemma) to move from

$$\vartheta_{\gamma_s \gamma_t} = t - s + o(t - s)$$

to

$$\delta \gamma_{st}^3 = - \int_s^t \gamma_r^2 \, d\gamma_r^1 + \int_s^t \gamma_r^1 \, d\gamma_r^2 + (t - s).$$
“Horizontal equation” ⇒ solutions to the LSDE satisfy $t \mapsto F(\gamma_t)$ constant.

**Theorem (surjectivity)**

Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h$. Let $\gamma : I \to \mathbb{H}$ solve the LSDE with $\gamma_0 = p$. Then, there exists $\varepsilon > 0$ such that

$$F^{-1}(F(p)) \cap B_\varepsilon(p) = \gamma(I) \cap B_\varepsilon(p).$$

No need of $C^{1,\alpha}_h$ (but we do not know how to get existence...)

**Proof** is a combination of two lemmas:

- “Horizontal injectivity” (due to non degeneracy of $p$) ⇒ we attach a region of injectivity (for the level set) at every $\gamma_t$;
- “Smart choice of $t$” As $t$ varies, such regions at $\gamma_t$ cover a neighbourhood of $p$. 
Surjectivity of solutions (on the level set)

“Horizontal equation” \( \Rightarrow \) solutions to the LSDE satisfy \( t \mapsto F(\gamma_t) \) constant.

**Theorem (surjectivity)**

Let \( p \) be non degenerate for \( F : \mathbb{H} \to \mathbb{R}^2, F \in C^1_h \). Let \( \gamma : I \to \mathbb{H} \) solve the LSDE with \( \gamma_0 = p \). Then, there exists \( \varepsilon > 0 \) such that

\[
F^{-1}(F(p)) \cap B_{\varepsilon}(p) = \gamma(I) \cap B_{\varepsilon}(p).
\]

No need of \( C^{1, \alpha}_h \) (but we do not know how to get existence. . .)

**Proof** is a combination of two lemmas:

- “Horizontal injectivity” (due to non degeneracy of \( p \)) \( \Rightarrow \) we attach a region of injectivity (for the level set) at every \( \gamma_t \);
- “Smart choice of \( t \)” As \( t \) varies, such regions at \( \gamma_t \) cover a neighbourhood of \( p \).
"Horizontal equation" ⇒ solutions to the LSDE satisfy $t \mapsto F(\gamma_t)$ constant.

**Theorem (surjectivity)**

Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h$. Let $\gamma : I \to \mathbb{H}$ solve the LSDE with $\gamma_0 = p$. Then, there exists $\epsilon > 0$ such that

$$F^{-1}(F(p)) \cap B_\epsilon(p) = \gamma(I) \cap B_\epsilon(p).$$

No need of $C^1_{h,\alpha}$ (but we do not know how to get existence...)

**Proof** is a combination of two lemmas:

- "Horizontal injectivity" (due to non degeneracy of $p$) ⇒ we attach a region of injectivity (for the level set) at every $\gamma_t$;
- "Smart choice of $t$" As $t$ varies, such regions at $\gamma_t$ cover a neighbourhood of $p$. 
Surjectivity of solutions (on the level set)

“Horizontal equation” ⇒ solutions to the LSDE satisfy $t \mapsto F(\gamma_t)$ constant.

Theorem (surjectivity)

Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h$. Let $\gamma : I \to \mathbb{H}$ solve the LSDE with $\gamma_0 = p$. Then, there exists $\varepsilon > 0$ such that

$$F^{-1}(F(p)) \cap B_\varepsilon(p) = \gamma(I) \cap B_\varepsilon(p).$$

No need of $C^{1,\alpha}_h$ (but we do not know how to get existence...)

Proof is a combination of two lemmas:

- “Horizontal injectivity” (due to non degeneracy of $p$) ⇒ we attach a region of injectivity (for the level set) at every $\gamma_t$;
- “Smart choice of $t$” As $t$ varies, such regions at $\gamma_t$ cover a neighbourhood of $p$. 
“Horizontal equation” ⇒ solutions to the LSDE satisfy $t \mapsto F(\gamma_t)$ constant.

**Theorem (surjectivity)**

Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h$. Let $\gamma : I \to \mathbb{H}$ solve the LSDE with $\gamma_0 = p$. Then, there exists $\varepsilon > 0$ such that

$$F^{-1}(F(p)) \cap B_\varepsilon(p) = \gamma(I) \cap B_\varepsilon(p).$$

No need of $C^{1,\alpha}_h$ (but we do not know how to get existence...)

**Proof** is a combination of two lemmas:

- “Horizontal injectivity” (due to non degeneracy of $p$) ⇒ we attach a region of injectivity (for the level set) at every $\gamma_t$;

- “Smart choice of $t$” As $t$ varies, such regions at $\gamma_t$ cover a neighbourhood of $p$. 
Lemma “Horizontal injectivity”

Let $\nabla_h F(p)$ be invertible. There exists $\varepsilon > 0$ such that, if $x, y \in B_\varepsilon(p)$,

$$F(x) = F(y) \quad \text{and} \quad |\vartheta_{xy}|^{1/2} \leq |y^1 - x^1| + |y^2 - x^2| \quad \Rightarrow \quad x = y.$$  

Proof: Horizontal Taylor expansion $\Rightarrow \nabla_h F(x)(y^1 - x^1, y^2 - x^2) = o(d(x, y))$

Lemma “Smart choice of $t$”

If $\varepsilon > 0$ is small enough, for every $x \in B_\varepsilon(p)$, there exists $t \in I$ with

$$|\vartheta_{\gamma s \gamma t}|^{1/2} \leq |\delta_{\gamma}^1| + |\delta_{\gamma}^2|$$

Proof: Use the “vertical” equation, $\vartheta_{\gamma s \gamma t} = t - s + o(t - s)$. 

Lemma “Horizontal injectivity”

Let $\nabla_h F(p)$ be invertible. There exists $\varepsilon > 0$ such that, if $x, y \in B_\varepsilon(p)$,

$$F(x) = F(y) \quad \text{and} \quad |\vartheta_{xy}|^{1/2} \leq |y^1 - x^1| + |y^2 - x^2| \quad \Rightarrow \quad x = y.$$

Proof: Horizontal Taylor expansion $\Rightarrow \nabla_h F(x)(y^1 - x^1, y^2 - x^2) = o(d(x, y))$

Lemma “Smart choice of $t$”

If $\varepsilon > 0$ is small enough, for every $x \in B_\varepsilon(p)$, there exists $t \in I$ with

$$|\vartheta_{s\gamma t}|^{1/2} \leq |\delta \gamma^1_{st}| + |\delta \gamma^2_{st}|$$

Proof: Use the “vertical” equation, $\vartheta_{s\gamma t} = t - s + o(t - s)$. 
**Lemma “Horizontal injectivity”**

Let $\nabla_h F(p)$ be invertible. There exists $\varepsilon > 0$ such that, if $x, y \in B_\varepsilon(p)$,

$$F(x) = F(y) \quad \text{and} \quad |\vartheta_{xy}|^{1/2} \leq |y^1 - x^1| + |y^2 - x^2| \quad \Rightarrow \quad x = y.$$  

**Proof:** Horizontal Taylor expansion $\Rightarrow \nabla_h F(x)(y^1 - x^1, y^2 - x^2) = o(d(x, y))$

**Lemma “Smart choice of $t$”**

If $\varepsilon > 0$ is small enough, for every $x \in B_\varepsilon(p)$, there exists $t \in I$ with

$$|\vartheta_{s\gamma t}|^{1/2} \leq \left| \delta\gamma_{st}^1 \right| + \left| \delta\gamma_{st}^2 \right|$$

**Proof:** Use the “vertical” equation, $\vartheta_{s\gamma t} = t - s + o(t - s)$. 
**Lemma “Horizontal injectivity”**

Let $\nabla_h F(p)$ be invertible. There exists $\varepsilon > 0$ such that, if $x, y \in B_\varepsilon(p)$,

$$F(x) = F(y) \quad \text{and} \quad |\vartheta_{xy}|^{1/2} \leq \left|y^1 - x^1\right| + \left|y^2 - x^2\right| \Rightarrow x = y.$$

**Proof:** Horizontal Taylor expansion $\Rightarrow \nabla_h F(x)(y^1 - x^1, y^2 - x^2) = o(d(x, y))$

**Lemma “Smart choice of $t$”**

If $\varepsilon > 0$ is small enough, for every $x \in B_\varepsilon(p)$, there exists $t \in I$ with

$$|\vartheta_{\gamma s \gamma t}|^{1/2} \leq \left|\delta \gamma^1_{st}\right| + \left|\delta \gamma^2_{st}\right|$$

**Proof:** Use the “vertical” equation, $\vartheta_{\gamma s \gamma t} = t - s + o(t - s)$. 
Lemma (Local uniqueness)

Any two solutions $\gamma, \tilde{\gamma}$ to the LSDE with $\gamma_0 = \tilde{\gamma}_0 = p$ coincide on a neighbourhood of $t = 0$.

Proof: Since both $\gamma, \tilde{\gamma}$, parametrize $F^{-1}(F(p))$, one has

$$\gamma_t = \tilde{\gamma}_{\varphi}(t).$$

The “vertical equation” gives

$$t - s + o(t - s) = \varphi_{\gamma_s \gamma_t} = \varphi_{\tilde{\gamma}_{\varphi(s)} \tilde{\gamma}_{\varphi(t)}} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t \Rightarrow$

$$\frac{d\varphi}{dt} = 1 \Rightarrow \varphi(t) = t.$$
Uniqueness of solutions

**Lemma (Local uniqueness)**

Any two solutions $\gamma, \bar{\gamma}$ to the LSDE with $\gamma_0 = \bar{\gamma}_0 = p$ coincide on a neighbourhood of $t = 0$.

**Proof:** Since both $\gamma, \bar{\gamma}$, parametrize $F^{-1}(F(p))$, one has

$$\gamma_t = \bar{\gamma}_{\varphi(t)}.$$

The “vertical equation” gives

$$t - s + o(t - s) = \vartheta_{\gamma s \gamma_t} = \vartheta_{\bar{\gamma}_{\varphi(s)} \bar{\gamma}_{\varphi(t)}} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t \Rightarrow$

$$\frac{d\varphi}{dt} = 1 \Rightarrow \varphi(t) = t.$$
Lemma (Local uniqueness)

Any two solutions $\gamma, \tilde{\gamma}$ to the LSDE with $\gamma_0 = \tilde{\gamma}_0 = p$ coincide on a neighbourhood of $t = 0$.

Proof: Since both $\gamma, \tilde{\gamma}$, parametrize $F^{-1}(F(p))$, one has

$$\gamma_t = \tilde{\gamma}_{\varphi(t)}.$$ 

The “vertical equation” gives

$$t - s + o(t - s) = \varphi_{\gamma(t)} = \varphi_{\tilde{\gamma}_{\varphi(t)}} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t$ \Rightarrow

$$\frac{d\varphi}{dt} = 1 \quad \Rightarrow \quad \varphi(t) = t.$$
Lemma (Local uniqueness)

Any two solutions $\gamma, \bar{\gamma}$ to the LSDE with $\gamma_0 = \bar{\gamma}_0 = p$ coincide on a neighbourhood of $t = 0$.

Proof: Since both $\gamma$, $\bar{\gamma}$, parametrize $F^{-1}(F(p))$, one has

$$\gamma_t = \bar{\gamma}_\varphi(t).$$

The “vertical equation” gives

$$t - s + o(t - s) = \vartheta \gamma_s \gamma_t = \vartheta \bar{\gamma}_{\varphi(s)} \bar{\gamma}_{\varphi(t)} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t \Rightarrow$

$$\frac{d\varphi}{dt} = 1 \quad \Rightarrow \quad \varphi(t) = t.$$
Lemma (Local uniqueness)

Any two solutions $\gamma, \bar{\gamma}$ to the LSDE with $\gamma_0 = \bar{\gamma}_0 = p$ coincide on a neighbourhood of $t = 0$.

Proof: Since both $\gamma, \bar{\gamma}$, parametrize $F^{-1}(F(p))$, one has

$$\gamma_t = \bar{\gamma}_\varphi(t).$$

The “vertical equation” gives

$$t - s + o(t - s) = \vartheta_{\gamma_s \gamma_t} = \vartheta_{\bar{\gamma}_\varphi(s) \bar{\gamma}_\varphi(t)} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t \Rightarrow$

$$\frac{d\varphi}{dt} = 1 \quad \Rightarrow \quad \varphi(t) = t.$$
## Theorem (Area formula)

Let $\gamma : I \to \mathbb{H}$ solve the LSDE. Then, for every interval $[a, b] \subseteq I$,

$$S^2(\gamma([a, b])) = \mathcal{L}^1([a, b]).$$

Actually we prove a more general Area formula for nice “vertical curves”.

## Theorem (Coarea formula)

Let $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h, \alpha$. Then for $A \subseteq \mathbb{H}$,

$$\int_A J_h F \, d\mathcal{L}^3 = \int_{\mathbb{R}^2} S^2 \left( A \cap F^{-1}(z) \right) \, d\mathcal{L}^2(z).$$

Proof uses area formula and blow-up argument. (Case $\alpha = 0$ is open).
Examples of level sets

Part of our arguments together with Whitney extension theorem $\rightarrow$ examples of $F \in C_{h}^{1,\alpha}$ with “bad” level sets.

**Theorem (Whitney (Vodopyanov ’06))**

Let $K \subseteq \mathbb{H}$ be compact, $\alpha \in (0, 1)$ and $F : K \rightarrow \mathbb{R}^2$, $F' : K \rightarrow \mathbb{R}^{2 \times 2}$ with

$$|F(x) - F(y) - F'(x) \cdot (y^1 - x^1, y^2 - x^2)| \leq cd(x, y)^{1+\alpha}$$

$$|F'(y) - F'(x)| \leq cd(x, y)^{\alpha}.$$

Then there an extension $F \in C^{1,\alpha} - h$ such that $F'(x) = \nabla_{h}F(x)$ for $x \in K$.

**Strategy:** for $(\eta^1, \eta^2) : I \rightarrow \mathbb{R}^2 \frac{1+\alpha}{2}$-Hölder, “lift” $\eta^3$ such that

$$\vartheta_{\eta^3} = t - s + o(t - s).$$

Then $K = \eta(I) F(x) = 0$ and $F'(x) = \text{Id.}$
Examples of level sets

Part of our arguments together with Whitney extension theorem $\rightarrow$ examples of $F \in C^1_{h,\alpha}$ with “bad” level sets.

**Theorem (Whitney (Vodopyanov ’06))**

Let $K \subseteq \mathbb{H}$ be compact, $\alpha \in (0, 1)$ and $F : K \rightarrow \mathbb{R}^2$, $F' : K \rightarrow \mathbb{R}^{2 \times 2}$ with

$$
|F(x) - F(y) - F'(x) \cdot (y^1 - x^1, y^2 - x^2)| \leq c d(x, y)^{1+\alpha}
$$

$$
|F'(y) - F'(x)| \leq c d(x, y)^{\alpha}.
$$

Then there an extension $F \in C^{1,\alpha} - h$ such that $F'(x) = \nabla_h F(x)$ for $x \in K$.

**Strategy:** for $(\eta^1, \eta^2) : I \rightarrow \mathbb{R}^2$ $\frac{1+\alpha}{2}$-Hölder, “lift” $\eta^3$ such that

$$
\dot{\eta}^3_{\eta^1} = t - s + o(t - s).
$$

Then $K = \eta(I)$ $F(x) = 0$ and $F'(x) = \text{Id}$. 
Examples of level sets

Part of our arguments together with Whitney extension theorem → examples of $F \in \mathcal{C}^{1,\alpha}_h$ with “bad” level sets.

**Theorem (Whitney (Vodopyanov ’06))**

Let $K \subseteq \mathbb{H}$ be compact, $\alpha \in (0, 1)$ and $F : K \to \mathbb{R}^2$, $F' : K \to \mathbb{R}^{2 \times 2}$ with

\[ |F(x) - F(y) - F'(x) \cdot (y^1 - x^1, y^2 - x^2)| \leq c d(x, y)^{1+\alpha} \]

\[ |F'(y) - F'(x)| \leq c d(x, y)^{\alpha}. \]

Then there an extension $F \in \mathcal{C}^{1,\alpha} - h$ such that $F'(x) = \nabla_h F(x)$ for $x \in K$.

**Strategy:** for $(\eta^1, \eta^2) : I \to \mathbb{R}^2$ $\frac{1+\alpha}{2}$-Hölder, “lift” $\eta^3$ such that

\[ \psi_{\eta^3_i, \eta^3_i} = t - s + o(t - s). \]

Then $K = \eta(I) F(x) = 0$ and $F'(x) = \text{Id.}$
To check that condition
\[ |F(x) - F(y) - F'(x) \cdot (y^1 - x^1, y^2 - x^2)| \leq cd(x, y)^{1+\alpha} \]
holds \( \rightarrow x = \eta_s, \ y = \eta_t \)

\[ |\delta \eta^1_{st}| + |\delta \eta^2_{st}| \leq cd(\eta_s, \eta_t)^{1+\alpha}. \]

Since \( 1 + \alpha > 1 \) and we argue on a small interval it is equivalent to prove
\[ |\delta \eta^1_{st}| + |\delta \eta^2_{st}| \leq c|\vartheta_{\eta_s \eta_t}|^{\frac{1+\alpha}{2}}, \]
which is satisfied because \((\eta^1, \eta^2)\) are \(\frac{1+\alpha}{2}\)-Hölder continuous and
\[ |t - s| \leq c|t - s + o(t - s)| = c|\vartheta_{\eta_s \eta_t}|. \]
To check that condition
\[ |F(x) - F(y) - F'(x) \cdot (y^1 - x^1, y^2 - x^2)| \leq c d(x, y)^{1+\alpha} \]
holds \( x = \eta_s, y = \eta_t \)
\[ |\delta \eta_{st}^1| + |\delta \eta_{st}^2| \leq c d(\eta_s, \eta_t)^{1+\alpha}. \]
Since \( 1 + \alpha > 1 \) and we argue on a small interval it is equivalent to prove
\[ |\delta \eta_{st}^1| + |\delta \eta_{st}^2| \leq c |\vartheta_{\eta_s \eta_t}|^{\frac{1+\alpha}{2}}, \]
which is satisfied because \((\eta^1, \eta^2)\) are \(\frac{1+\alpha}{2}\)-Hölder continuous and
\[ |t - s| \leq c |t - s + o(t - s)| = c |\vartheta_{\eta_s \eta_t}|. \]
To check that condition
\[
\left| F(x) - F(y) - F'(x) \cdot (y^1 - x^1, y^2 - x^2) \right| \leq c d(x, y)^{1+\alpha}
\]
holds \( x = \eta_s, \ y = \eta_t \)

\[
|\delta \eta^1_{st}| + |\delta \eta^2_{st}| \leq c d(\eta_s, \eta_t)^{1+\alpha}.
\]
Since \( 1 + \alpha > 1 \) and we argue on a small interval it is equivalent to prove

\[
|\delta \eta^1_{st}| + |\delta \eta^2_{st}| \leq c |\vartheta \eta_s \eta_t|^{\frac{1+\alpha}{2}},
\]
which is satisfied because \((\eta^1, \eta^2)\) are \(\frac{1+\alpha}{2}\)-Hölder continuous and

\[
|t - s| \leq c |t - s + o(t - s)| = c |\vartheta \eta_s \eta_t|.
Can we produce examples of higher dimension? Consider the case $F : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}^4$, so that we expect a 2-dimensional surface $\varphi_s = \varphi_{s^1,s^2}$. Notation:

$$(x, \tilde{x}) = (x^1, x^2, x^3, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \mathbb{H} \times \mathbb{H}$$

contact forms $\vartheta$ and $\tilde{\vartheta}$

$$\vartheta_{xy} := (y^3 - x^3) + x^2(y^1 - x^1) - x^1(y^2 - x^2).$$

Problem: analogue of the “vertical” condition $\vartheta_{\eta s \eta t} = t - s + o(t - s)$?

$$
\begin{pmatrix}
\vartheta_{\eta s \eta t} \\
\tilde{\vartheta}_{\eta s \eta t}
\end{pmatrix} =
\begin{pmatrix}
a^{11}_s & a^{12}_s \\
a^{21}_s & a^{22}_s
\end{pmatrix}
\begin{pmatrix}
t^1 - s^1 \\
t^2 - s^2
\end{pmatrix} + o(t - s),
$$

How to integrate it?

Possible approach: extend calculus to

1. “rough” differential forms (R. Zust → Young case)
2. solve exterior differential systems (Frobenius theorem)
Can we produce examples of higher dimension? Consider the case $F : \mathbb{H} \times \mathbb{H} \to \mathbb{R}^4$, so that we expect a 2-dimensional surface $\varphi_s = \varphi_{s^1, s^2}$.

Notation:

$$(x, \tilde{x}) = (x^1, x^2, x^3, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \mathbb{H} \times \mathbb{H}$$

contact forms $\vartheta$ and $\tilde{\vartheta}$

$$\vartheta_{xy} := (y^3 - x^3) + x^2(y^1 - x^1) - x^1(y^2 - x^2).$$

Problem: analogue of the “vertical” condition $\vartheta_{\eta_s \eta_t} = t - s + o(t - s)$?

$$
\begin{pmatrix}
\vartheta_{\eta_s \eta_t} \\
\tilde{\vartheta}_{\eta_s \eta_t}
\end{pmatrix} = 
\begin{pmatrix}
a_{s}^{11} & a_{s}^{12} \\
a_{s}^{21} & a_{s}^{22}
\end{pmatrix} 
\begin{pmatrix}
t^1 - s^1 \\
t^2 - s^2
\end{pmatrix} + o(t - s),
$$

How to integrate it?

Possible approach: extend calculus to

1. “rough” differential forms (R. Zurst → Young case)
2. solve exterior differential systems (Frobenius theorem)
Can we produce examples of higher dimension? Consider the case $F : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}^4$, so that we expect a 2-dimensional surface $\varphi_s = \varphi_{s^1, s^2}$. Notation:

$$(x, \tilde{x}) = (x^1, x^2, x^3, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \mathbb{H} \times \mathbb{H}$$

contact forms $\vartheta$ and $\tilde{\vartheta}$

$$\vartheta_{xy} := (y^3 - x^3) + x^2(y^1 - x^1) - x^1(y^2 - x^2).$$

**Problem:** analogue of the “vertical” condition $\vartheta_{\eta_s \eta_t} = t - s + o(t - s)$?

$$
\begin{pmatrix}
\vartheta_{\eta_s \eta_t} \\
\tilde{\vartheta}_{\eta_s \eta_t}
\end{pmatrix}
= 
\begin{pmatrix}
a_s^{11} & a_s^{12} \\
a_s^{21} & a_s^{22}
\end{pmatrix}
\begin{pmatrix}
t^1 - s^1 \\
t^2 - s^2
\end{pmatrix} + o(t - s),
$$

How to integrate it?

Possible approach: extend calculus to

1. “rough” differential forms (R. Zust $\rightarrow$ Young case)
2. solve exterior differential systems (Frobenius theorem)
R. Zust (2010) showed that the integral of a $k$-form on any cube $Q \subseteq \mathbb{R}^k$

$$\int_Q f \, dg^1 \wedge dg^2 \wedge \ldots \wedge dg^k$$

is “robustly” defined (continuous limit w.r.t. approximations) if

$$f \in C^\alpha, \, g^1 \in C^{\beta_1}, \ldots, \, g^k \in C^{\beta_k} \quad \text{with} \quad \alpha + \beta_1 + \ldots + \beta_k > k.$$ 

(Joint with E. Stepanov) generalize his result as a sewing lemma for $k$-forms

Examples from stochastic analysis? (Brownian sheets, . . . )
R. Zust (2010) showed that the integral of a $k$-form on any cube $Q \subseteq \mathbb{R}^k$

$$\int_Q f \, dg^1 \wedge dg^2 \wedge \ldots \wedge dg^k$$

is “robustly” defined (continuous limit w.r.t. approximations) if

$$f \in C^\alpha, \, g^1 \in C^{\beta_1}, \ldots, \, g^k \in C^{\beta_k} \quad \text{with} \quad \alpha + \beta_1 + \ldots + \beta_k > k.$$

(Joint with E. Stepanov) generalize his result as a sewing lemma for $k$-forms

Examples from stochastic analysis? (Brownian sheets, …)
R. Zust (2010) showed that the integral of a $k$-form on any cube $Q \subseteq \mathbb{R}^k$

$$\int_Q f \, dg^1 \wedge dg^2 \wedge \ldots \wedge dg^k$$

is “robustly” defined (continuous limit w.r.t. approximations) if

$$f \in C^\alpha, g^1 \in C^{\beta_1}, \ldots, g^k \in C^{\beta_k} \quad \text{with } \alpha + \beta_1 + \ldots + \beta_k > k.$$

(Joint with E. Stepanov) generalize his result as a \textit{sewing lemma} for $k$-forms

Examples from stochastic analysis? (Brownian sheets,\ldots)
A Frobenius theorem in Hölder classes

The (Euclidean) implicit function for $F : \mathbb{R}^n \to \mathbb{R}^k$ can be seen as an instance of Frobenius theorem, for systems of differential equations.

Another example is the “Pfaff” system for a parametrized surface $\varphi(s) = \varphi(s^1, s^2)$

\[
\begin{align*}
\partial_{s^1} \varphi_s &= \sum_{i=1}^{n} f^i(s, \varphi_s) \partial_{s^1} g^i_s \\
\partial_{s^2} \varphi_s &= \sum_{i=1}^{n} f^i(s, \varphi_s) \partial_{s^2} g^i_s,
\end{align*}
\]

or equivalently

\[
\delta \varphi_{st} = \sum_{i=1}^{n} f^i(s, \eta_s) \delta g^i_{st} + o(t - s).
\]

Problem: formulate (necessary) and sufficient conditions to be well-posed.

Partial positive results. In particular, $g$ must be more than $\frac{2}{3}$-Hölder continuous.
A Frobenius theorem in Hölder classes

The (Euclidean) implicit function for $F : \mathbb{R}^n \to \mathbb{R}^k$ can be seen as an instance of Frobenius theorem, for systems of differential equations.

Another example is the “Pfaff” system for a parametrized surface $\varphi(s) = \varphi(s^1, s^2)$

\[
\begin{align*}
\partial_{s^1} \varphi_s &= \sum_{i=1}^{n} f^i(s, \varphi_s) \partial_{s^1} g^i_s \\
\partial_{s^2} \varphi_s &= \sum_{i=1}^{n} f^i(s, \varphi_s) \partial_{s^2} g^i_s,
\end{align*}
\]

or equivalently

\[
\delta \varphi_{st} = \sum_{i=1}^{n} f^i(s, \eta_s) \delta g^i_{st} + o(t - s).
\]

Problem: formulate (necessary) and sufficient conditions to be well-posed.

Partial positive results. In particular, $g$ must be more than $\frac{2}{3}$-Hölder continuous.
A Frobenius theorem in Hölder classes

The (Euclidean) implicit function for $F : \mathbb{R}^n \to \mathbb{R}^k$ can be seen as an instance of Frobenius theorem, for systems of differential equations.

Another example is the “Pfaff” system for a parametrized surface $\varphi(s) = \varphi(s^1, s^2)$

\[
\begin{align*}
\partial_{s^1} \varphi_s &= \sum_{i=1}^n f^i(s, \varphi_s) \partial_{s^1} g^i_s \\
\partial_{s^2} \varphi_s &= \sum_{i=1}^n f^i(s, \varphi_s) \partial_{s^2} g^i_s,
\end{align*}
\]

or equivalently

\[
\delta \varphi_{st} = \sum_{i=1}^n f^i(s, \eta_s) \delta g^i_{st} + o(t - s).
\]

Problem: formulate (necessary) and sufficient conditions to be well-posed.

Partial positive results. In particular, $g$ must be more than $\frac{2}{3}$-Hölder continuous.
The (Euclidean) implicit function for $F : \mathbb{R}^n \to \mathbb{R}^k$ can be seen as an instance of Frobenius theorem, for systems of differential equations.

Another example is the “Pfaff” system for a parametrized surface

$$\varphi(s) = \varphi(s^1, s^2)$$

\[
\begin{aligned}
\partial_{s^1} \varphi_s &= \sum_{i=1}^{n} f^i(s, \varphi_s) \partial_{s^1} g^i_s \\
\partial_{s^2} \varphi_s &= \sum_{i=1}^{n} f^i(s, \varphi_s) \partial_{s^2} g^i_s,
\end{aligned}
\]

or equivalently

$$\delta \varphi_{st} = \sum_{i=1}^{n} f^i(s, \eta_s) \delta g^i_{st} + o(t - s).$$

**Problem:** formulate (necessary) and sufficient conditions to be well-posed.

Partial positive results. In particular, $g$ must be more than $\frac{2}{3}$-Hölder continuous.
Further open problems

- Relax $\alpha > 0$ condition:
  - compactness as $\alpha \to 0$,
  - a.e. level set?
  - other notions of integrals?

- Splitting case $F : \mathbb{H} \to \mathbb{R}$ – no need of integrals!
Further open problems

- Relax \( \alpha > 0 \) condition:
  - compactness as \( \alpha \to 0 \),
  - a.e. level set?
  - other notions of integrals?

- Splitting case \( F : \mathbb{H} \to \mathbb{R} \) – no need of integrals!