# On level sets in the Heisenberg group

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# An implicit function theorem on Lie groups

## Implicit function theorem

Regular level sets of a $C^1$ map between Euclidean spaces have a

local $C^1$ parametrization.

## Problem

What happens if we replace Euclidean spaces with more general Lie groups?

We study the simplest non-trivial case: $F : \mathbb{H} \approx \mathbb{R}^3 \to \mathbb{R}^k$.

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# Structure of the talk

# Heisenberg group: differential structure

The group $\mathbb{H}$ is a non-commutative Lie group, with two generators.

- On $\mathbb{H} = \mathbb{R}^3$, $x = (x^1, x^2, x^3)$, consider the two (horizontal) vector fields

$$X_1(x) := \partial_1 - x^2\partial_3 \qquad X_2(x) := \partial_2 + x^1\partial_3$$

$$[X_1, X_2] = [\partial_1 - x^2\partial_3, \partial_2 + x^1\partial_3] = 2\partial_3 \quad \text{(Hörmander condition).}$$

- Dual description: contact 1-form

$$\theta = dx^3 + x^2dx^1 - x^1dx^2 \quad \Rightarrow \quad d\theta = -2dx^1 \wedge \mathrm{d}x^2$$

- Horizontal tangent at $x \in \mathbb{H}$ is span $\{X_1(x), X_2(x)\} = \mathrm{Ker}\,\theta_x$.

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# Heisenberg group: curves and distance

- A (smooth) curve $\eta : I \to \mathbb{H}$ is horizontal if, for $t \in I$,

$$\theta_{\eta_t}(\dot{\eta}_t) = \dot{\eta}^3_t + \eta^2_t \dot{\eta}^1_t - \eta^1_t \dot{\eta}^2_t = 0.$$

- Imposing $X_1(x)$ $X_2(x)$ are orthonormal $\Rightarrow$ CC-distance

$$d(x, y) := \inf \left\{ \int_0^1 |\dot{\eta}_t| : \ \eta \text{ horizontal}, \eta_0 = x, \eta_1 = y. \right\}$$

- Equivalence

$$d(x, y) \approx |y^1 - x^1| + |y^2 - x^2| + |\vartheta_{xy}|^{1/2},$$

where a “discrete” contact form appears

$$\vartheta_{xy} := (y^3 - x^3) + x^2(y^1 - x^1) - x^1(y^2 - x^2).$$

(Recall $\theta = \mathrm{d}x^3 + x^2\,\mathrm{d}x^1 - x^1\,\mathrm{d}x^2$).

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# Examples

1. Horizontal curve $\leftrightarrow$

$$\eta_t^3 - \eta_s^3 = \int_s^t \eta_r^2 \dot{\eta}_r^1 \, dr - \int_r^r \eta_r^1 \dot{\eta}_r^2 \, dr.$$

   If $(\eta_t^1, \eta_t^2) = (\eta_s^1, \eta_s^2) \to \eta_t^3 - \eta_s^3 =$ signed area.

2. If $(\eta^1, \eta^2)$ are $\frac{1+\alpha}{2}$-Hölder continuous $\to$ horizontal lift

$$\theta_{\eta_t}(\dot{\eta}_t) = 0$$

   in the sense Young integrals or in the “incremental” sense

$$\vartheta_{\eta_s \eta_t} = (\eta_t^3 - \eta_s^3) + \eta_s^2(\eta_t^1 - \eta_s^1) - \eta_s^1(\eta_t^2 - \eta_s^2) = o(t-s)$$

3. If $\alpha = 0$, pure area rough path

$$n^{-1/2}(\cos(nt), \sin(nt)) \quad n \to \infty.$$

   The limit of horizontal lifts is not horizontal!

# Heisenberg group: regular maps

- We “measure” regularity of $F : \mathbb{H} \to \mathbb{R}^k$ in terms of horizontal derivatives

$$\nabla_h F(x) := (X_1 F(x), X_2 F(x)) \,.$$

  $p \in \mathbb{H}$ is non degenerate for $F$ if $\nabla_h F(p)$ has maximum rank

- For $\alpha \in (0,1)$, $F \in C_h^{1,\alpha}$ if $x \mapsto \nabla_h F(x)$ is (well-defined and) $\alpha$-Hölder continuous, (w.r.t. $d$). ($F \in C_h^1$ if just continuous).

- Fact: There are $F \in C^{1,\alpha}$ nowhere (Euclidean) differentiable on a set of positive Lebesgue measure.

**Problem**

Locally parametrize $F^{-1}(F(p))$ for $F \in C_h^1$ for non degenerate $p$'s.

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# Literature on level sets in Heisenberg group, $F : \mathbb{H} \to \mathbb{R}^k$

$k = 1$: algebraic splitting phenomenon (Ambrosio-SerraCassano-Vittone, . . . ) $\Rightarrow$ "intrinsic graphs", parametrized surfaces via group operation. (Interesting connection with non-linear PDE's, recall talk by Katrin Fässler).

$k = 2$: Magnani-Leonardi (2010) $\Rightarrow$ continuous curves, intersections of two intrinsic surfaces.

$k = 2$: Kozhevnikov (2011) $\Rightarrow$ $\beta$-Hölder continuous curves ($\beta < 1/2$) via a sub-Riemannian Reifenberg-type argument.

For $k = 2$, parametrizations are quite implicit: is a "good calculus" missing?

**Main results (Magnani-Stepanov-T., 2016): $k = 2$.**

Explicit "Level Set Differential Equation" (LSDE).

- Prove existence, uniqueness, and stability w.r.t. approximations for $F \in C_h^{1,\alpha}$ ($\alpha > 0$) using tools from Young integration (Rough paths).
- Prove area formula and (re)-obtain a coarea formula for $F \in C_h^{1,\alpha}$.

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# The Euclidean ODE argument

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be $C^1$.

- Write $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, 2, 3$,
- $F = (F^1, F^2)$ and

$$\nabla F = \begin{pmatrix} \partial_1 F^1 & \partial_2 F^1 & \partial_3 F^1 \\ \partial_1 F^2 & \partial_2 F^2 & \partial_3 F^2 \end{pmatrix} = (\nabla_{12} F, \nabla_3 F) \quad \text{with } \nabla_{12} F \text{ invertible.}$$

Differentiating $F(\gamma_t) = c$,

$$\begin{pmatrix} \dot\gamma_t^1 \\ \dot\gamma_t^2 \end{pmatrix} = -\left(\nabla_{12} F(\gamma_t)\right)^{-1} \nabla_3 F(\gamma_t) \dot\gamma_t^3 .$$

Impose $\dot\gamma_t^3 = 1$ and solve (Peano) for $(\gamma_t^1, \gamma_t^2)$. (Uniqueness of solutions?)

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# A naive approach to the LSDE

In $\mathbb{H} \sim \mathbb{R}^3$, recast the ODE

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in terms of the horizontal derivatives $X_1F$, $X_2F$ (change of coordinates):

$$\begin{pmatrix} \dot\gamma_t^1 \\ \dot\gamma_t^2 \end{pmatrix} = -\left(\nabla_h F(\gamma_t)\right)^{-1} \nabla_3 F(\gamma_t)\theta_{\gamma_t}(\dot\gamma_t).$$

In analogy with $\dot\gamma_t^3 = 1$, set $\theta_{\gamma_t}(\dot\gamma_t) = 1 \Rightarrow$ non-horizontal, (vertical), curve.

Two difficulties:

1. The "vertical derivative" $\nabla_3 F$ may not be defined, even if $F \in C^{1,\alpha}$ with $0 < \alpha < 1$.
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# Heuristics

Here's a "rule of thumb" to define

$$\begin{pmatrix} \dot\gamma_t^1 \\ \dot\gamma_t^2 \end{pmatrix} = -\left(\nabla_h F(\gamma_t)\right)^{-1} \nabla_3 F(\gamma_t)\theta_{\gamma_t}(\dot\gamma_t), \quad \theta_{\gamma_t}(\dot\gamma_t) = 1.$$

1. "Integrating" $\theta_{\gamma_t}(\dot\gamma_t) = 1$ gives $\vartheta_{\gamma_s\gamma_t} = t - s$,

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5. $\frac{1+\alpha}{2}$-Hölder continuity is consistent with assumption 1, closing the circle.

# Heuristics

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The equation

$$\theta_{\gamma_t}(\dot\gamma_t) = \dot\gamma^3 + \gamma^2_t\dot\gamma^1_t - \gamma^1_t\dot\gamma^2_t = 1$$

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Instead of “differentiating”, we use finite differences ⇒ horizontal Taylor expansion:

$$F(y) - F(x) - \nabla_h F(x) \begin{pmatrix} y^1 - x^1 \\ y^2 - x^2 \end{pmatrix} - \nabla_3 F(x) \vartheta_{xy} = R_{xy}.$$

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**Definition (LSDE)**

Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C_h^1$. We say $\gamma : I \to \mathbb{H}$ is a solution to the level set differential equation (LSDE) if it is continuous and

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for every $s, t \in I$.

- The "horizontal equation" yields that $t \mapsto F(\gamma_t)$ is constant.
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# Existence of solutions

**Theorem (Existence)**

Let $\alpha > 0$ and $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C_h^{1,\alpha}$. Then, there exists $\delta > 0$ and $\gamma : [-\delta, \delta] \to \mathbb{H}$ solving the LSDE with $\gamma_0 = p$.

- Proof via Leray-Schauder fixed point on a subset of $C^{\frac{1+\alpha}{2}}([-\delta, \delta]; \mathbb{R}^3)$.
- Need of $\alpha > 0$: use Young integral (sewing lemma) to move from

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# Surjectivity of solutions (on the level set)

"Horizontal equation" $\Rightarrow$ solutions to the LSDE satisfy $t \mapsto F(\gamma_t)$ constant.

**Theorem (surjectivity)**

Let $p$ be non degenerate for $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C^1_h$. Let $\gamma : I \to \mathbb{H}$ solve the LSDE with $\gamma_0 = p$. Then, there exists $\varepsilon > 0$ such that

$$F^{-1}(F(p)) \cap B_\varepsilon(p) = \gamma(I) \cap B_\varepsilon(p).$$

No need of $C^{1,\alpha}_h$ (but we do not know how to get existence...)

Proof is a combination of two lemmas:

- "Horizontal injectivity" (due to non degeneracy of $p$) $\Rightarrow$ we attach a region of injectivity (for the level set) at every $\gamma_t$;
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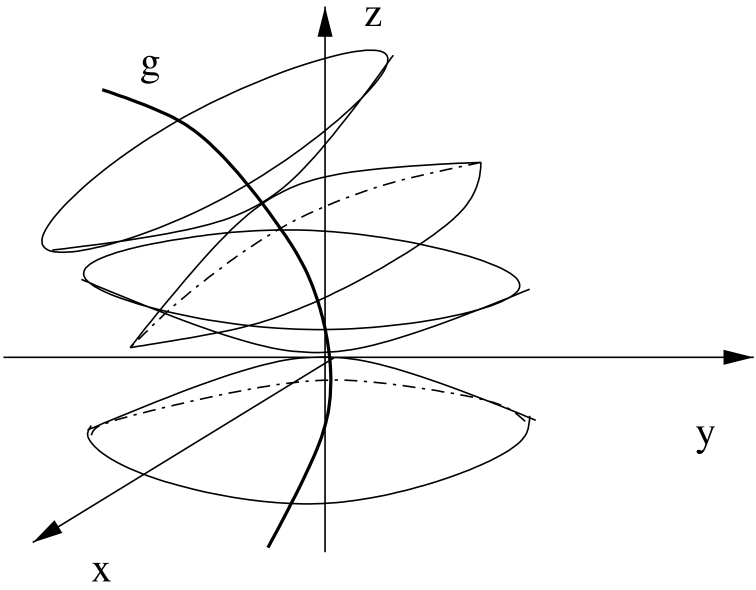
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**Lemma “Horizontal injectivity”**

Let $\nabla_h F(p)$ be invertible. There exists $\varepsilon > 0$ such that, if $x$, $y \in B_\varepsilon(p)$,

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$$|\vartheta_{\gamma_s \gamma_t}|^{1/2} \leq \left|\delta\gamma^1_{st}\right| + \left|\delta\gamma^2_{st}\right|$$

Proof: Use the "vertical" equation, $\vartheta_{\gamma_s \gamma_t} = t - s + o(t - s)$.

## Lemma “Horizontal injectivity”

Let $\nabla_h F(p)$ be invertible. There exists $\varepsilon > 0$ such that, if $x, y \in B_\varepsilon(p)$,

$$F(x) = F(y) \quad \text{and} \quad |\vartheta_{xy}|^{1/2} \leq \left|y^1 - x^1\right| + \left|y^2 - x^2\right| \quad \Rightarrow \quad x = y.$$

Proof: Horizontal Taylor expansion $\Rightarrow \nabla_h F(x)(y^1 - x^1, y^2 - x^2) = o\left(d(x, y)\right)$

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# Uniqueness of solutions

**Lemma (Local uniqueness)**

Any two solutions $\gamma$, $\bar{\gamma}$ to the LSDE with $\gamma_0 = \bar{\gamma}_0 = p$ coincide on a neighbourhood of $t = 0$.

Proof: Since both $\gamma$, $\bar{\gamma}$, parametrize $F^{-1}(F(p))$, one has

$$\gamma_t = \bar{\gamma}_{\varphi(t)}.$$

The "vertical equation" gives

$$t - s + o(t - s) = \vartheta_{\gamma_s \gamma_t} = \vartheta_{\bar{\gamma}_{\varphi(s)} \bar{\gamma}_{\varphi(t)}} = \varphi(t) + \varphi(s) + o(\varphi(t) - \varphi(s)).$$

Divide by $t - s$ and let $s \to t \Rightarrow$

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# Further properties

### Theorem (Area formula)

Let $\gamma : I \to \mathbb{H}$ solve the LSDE. Then, for every interval $[a, b] \subseteq I$,

$$\mathcal{S}^2(\gamma([a, b])) = \mathcal{L}^1([a, b]).$$

Actually we prove a more general Area formula for nice “vertical curves”.

### Theorem (Coarea formula)

Let $F : \mathbb{H} \to \mathbb{R}^2$, $F \in C_h^{1,\alpha}$. Then for $A \subseteq \mathbb{H}$,

$$\int_A J_h F \, d\mathscr{L}^3 = \int_{\mathbb{R}^2} \mathcal{S}^2 \left( A \cap F^{-1}(z) \right) d\mathscr{L}^2(z).$$

Proof uses area formula and blow-up argument. (Case $\alpha = 0$ is open).

# Examples of level sets

Part of our arguments together with Whitney extension theorem $\rightarrow$ examples of $F \in C_h^{1,\alpha}$ with "bad" level sets.

**Theorem (Whitney (Vodopyanov '06))**

*Let $K \subseteq \mathbb{H}$ be compact, $\alpha \in (0,1)$ and $F : K \to \mathbb{R}^2$, $F' : K \to \mathbb{R}^{2\times 2}$ with*

$$\left| F(x) - F(y) - F'(x) \cdot (y^1 - x^1, y^2 - x^2) \right| \leq c d(x,y)^{1+\alpha}$$

$$|F'(y) - F'(x)| \leq c d(x,y)^{\alpha}.$$

*Then there an extension $F \in C^{1,\alpha} - h$ such that $F'(x) = \nabla_h F(x)$ for $x \in K$.*

Strategy: for $(\eta^1, \eta^2) : I \to \mathbb{R}^2$ $\frac{1+\alpha}{2}$-Hölder, "lift" $\eta^3$ such that

$$\vartheta_{\eta_s^3 \eta_t^3} = t - s + o(t-s).$$

Then $K = \eta(I)$ $F(x) = 0$ and $F'(x) = \mathrm{Id}$.

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To check that condition

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holds $\rightarrow x = \eta_s$, $y = \eta_t$

$$|\delta\eta^1_{st}| + |\delta\eta^2_{st}| \le c\mathrm{d}(\eta_s, \eta_t)^{1+\alpha}.$$

Since $1+\alpha > 1$ and we argue on a small interval it is equivalent to prove

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# Lift of surfaces

Can we produce examples of higher dimension? Consider the case $F : \mathbb{H} \times \mathbb{H} \to \mathbb{R}^4$, so that we expect a 2-dimensional surface $\varphi_s = \varphi_{s^1,s^2}$.
Notation:

$$(x, \tilde{x}) = (x^1, x^2, x^3, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \mathbb{H} \times \mathbb{H}$$

contact forms $\vartheta$ and $\tilde{\vartheta}$

$$\vartheta_{xy} := (y^3 - x^3) + x^2(y^1 - x^1) - x^1(y^2 - x^2).$$

Problem: analogue of the "vertical" condition $\vartheta_{\eta_s \eta_t} = t - s + o(t - s)$?

$$\begin{pmatrix} \vartheta_{\eta_s \eta_t} \\ \tilde{\vartheta}_{\eta_s \eta_t} \end{pmatrix} = \begin{pmatrix} a_s^{11} & a_s^{12} \\ a_s^{21} & a_s^{22} \end{pmatrix} \begin{pmatrix} t^1 - s^1 \\ t^2 - s^2 \end{pmatrix} + o(t - s),$$

How to integrate it?
Possible approach: extend calculus to

1. "rough" differential forms (R. Zust $\to$ Young case)
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# A sewing lemma for differential forms

R. Zust (2010) showed that the integral of a $k$-form on any cube $Q \subseteq \mathbb{R}^k$

$$\int_Q f \, \mathrm{d}g^1 \wedge \mathrm{d}g^2 \wedge \ldots \wedge \mathrm{d}g^k$$

is "robustly" defined (continuous limit w.r.t. approximations) if

$$f \in C^{\alpha}, g^1 \in C^{\beta_1}, \ldots, g^k \in C^{\beta_k} \quad \text{with } \alpha + \beta_1 + \ldots + \beta_k > k.$$

(Joint with E. Stepanov) generalize his result as a sewing lemma for $k$-forms

Examples from stochastic analysis? (Brownian sheets,. . . )

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# A Frobenius theorem in Hölder classes

The (Euclidean) implicit function for $F : \mathbb{R}^n \to \mathbb{R}^k$ can be seen as an instance of Frobenius theorem, for systems of differential equations.

Another example is the "Pfaff" system for a parametrized surface $\varphi(s) = \varphi(s^1, s^2)$

$$\begin{cases} \partial_{s^1} \varphi_s & = \sum_{i=1}^n f^i(s, \varphi_s) \partial_{s^1} g^i_s \\ \partial_{s^2} \varphi_s & = \sum_{i=1}^n f^i(s, \varphi_s) \partial_{s^2} g^i_s, \end{cases}$$

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Partial positive results. In particular, $g$ must be more than $\frac{2}{3}$-Hölder continuous.

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# Further open problems

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