Random dynamical systems and rough paths

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Outline

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- Main tool: Multiplicative Ergodic Theorem; yields existence of Lyapunovexponents for linear systems (Oseledets '68).
- For non-linear systems, one may look at their linearization, and the MET can still provide information about the local behaviour of the non-linear system (stable manifold theorem, Ruelle '79).

Stable manifolds for SDE

Stable manifold theorem for SDE. (Mohammed-Scheutzow '99)
Assume

$$\mathrm{d}Y_t^\xi = b(Y_t^\xi)\,\mathrm{d}t + \sigma(Y_t^\xi)\circ\mathrm{d}B_t(\omega), \quad Y_0^\xi = \xi \in \mathbb{R}^m$$

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- Differential of the flow solves a linear equation; existence of Lyanpunovexponents follow from the MET (under certain assumptions):

$$\mathbb{P} - \lim_{t\to\infty} \frac{1}{t} \log \left| D_\xi Y_t^{\xi(\omega)} v \right| \in \{\lambda_l < \ldots < \lambda_1\},$$

$1 \leq l \leq m$, $v \in \mathbb{R}^m$.

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  $1 \leq l \leq m$, $v \in \mathbb{R}^m$.

- Possible to conclude existence of stable (or unstable) manifolds around stationary solutions for the original equation (following Ruelle's strategy).

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- Itō-theory not really necessary; only used to make sense of initial equation.
- Markov property used only to speak about invariant measures. In fact, not really necessary, too, more general concept of invariant measures available in the theory of RDS (cf. below).
- One of our goals: Build a bridge between the "two cultures" (L. Arnold) Dynamical systems and Stochastic analysis.

Long term goal: Prove more general stable manifold theorem for

$$\mathrm{d}Y_t = b(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}\mathbf{X}_t(\omega) \tag{1}$$

with $t \mapsto \mathbf{X}_t$ rough paths lift of a stochastic process (e.g. fBm).

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Remark: In general, solution Y to (1) not Markovian.

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Metric dynamical systems

Definition.

$(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ called *metric dynamical system* if

(i) $(\omega, t) \mapsto \theta_t \omega$ is measurable

(ii) $\theta_0 = \mathrm{Id}_\Omega$

(iii) $\theta_{t+s} = \theta_t \circ \theta_s$

(iv) $\mathbb{P} = \mathbb{P} \circ \theta_t^{-1}$ for all $t \in \mathbb{R}$.

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Example: $\Omega = \mathcal{C}_0(\mathbb{R}, \mathbb{R}^d)$, θ Wiener shift:

$$\theta_t\omega := \omega(t + \cdot) - \omega(t).$$

If (iv) holds, coordinate process has stationary increments.

Construction of MDS from processes with stationary increments

Assume that $\bar{X}$ is $\mathbb{R}^d$-valued process defined on some probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with $\bar{X}(\bar{\omega}) \in \mathcal{C}_0(\mathbb{R}, \mathbb{R}^d)$ for all $\bar{\omega} \in \bar{\Omega}$.

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- If $\bar{X}$ has stationary increments, can build MDS as follows:
  - $\Omega := \mathcal{C}_0(\mathbb{R}, \mathbb{R}^d)$
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  - $\mathcal{F} :=$ Borel sigma-algebra
  - $\mathbb{P} :=$ Law of $\bar{X}$
  - $\theta :=$ Wiener shift
- Coordinate process X on this space has same law as $\bar{X}$ and satisfies the cocycle (or helix) property:

$$X_{t+s}(\omega) - X_s(\omega) = X_t(\theta_s \omega).$$

Definition.
A *continuous random dynamical system* (RDS) on a topological space X is a metric DS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t))$ with mapping

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(ii) $\varphi(0,\omega,\cdot) = \mathrm{Id}_X$ for all $\omega \in \Omega$ and

$$\varphi(t+s,\omega,\cdot) = \varphi(t,\theta_s\omega,\cdot) \circ \varphi(s,\omega,\cdot) \quad \text{"cocycle property"}$$

for all $s,t \in [0,\infty)$ and all $\omega \in \Omega$.

Example: SDEs

Example: $\varphi(t,\omega,\xi) := \phi(0,t,\omega,\xi)$ where

$$\phi\colon [0,\infty[\times[0,\infty[\times\Omega\times\mathbb{R}^m \to \mathbb{R}^m$$

solution (semi-)flow to SDE

$$\mathrm{d}Y = b(Y)\,\mathrm{d}t + \sigma(Y)\circ\mathrm{d}B$$

if b,σ sufficiently "nice".

Cocycle property for Itō-SDEs

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- Not a problem for a pathwise calculus.

Rough path cocycles

Definition.

An MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t))$ together with a rough path valued process $\mathbf{X}$ defined on this probability space is called a *rough paths cocycle* if the cocycle relation

$$\mathbf{X}_{s+t}(\omega) = \mathbf{X}_s(\omega) \otimes \mathbf{X}_t(\theta_s \omega) \qquad (2)$$

holds for every s, t and $\omega \in \Omega$.

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- on the first level, property (2) reads

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RDEs induce RDS

Theorem (Bailleul, R., Scheutzow '17).

If $\mathbf{X}$ is a rough paths cocycle, the flow of the rough differential equation

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Proof.

Straightforward!

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If a geometric rough paths valued process $\bar{\mathbf{X}}$ has stationary increments, there is a rough paths cocycle $\mathbf{X}$ with the same law as $\bar{\mathbf{X}}$.

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As for $\mathbb{R}^d$-valued processes.

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Theorem (Bailleul, R., Scheutzow).

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Example.

fBm with Hurst parameter $H \in (1/4, 1/2]$.

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Invariant measures for RDS

Definition.
Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t), \varphi)$ be a RDS on a measurable space $(X, \mathcal{B})$. Then the mapping

$$\begin{aligned} \Theta_t \colon \Omega \times X &\to \Omega \times X \\ (\omega, x) &\mapsto (\theta_t \omega, \varphi(t, \omega, x)) \end{aligned}$$

is called *skew product*.
A probability measure μ on $\mathcal{F} \otimes \mathcal{B}$ is called *invariant* for φ if it has first marginal $\mathbb{P}$ and if $\mu \circ \Theta_t^{-1} = \mu$ for all $t \geq 0$.

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- Existence of invariant measures implies existence of a *stationary state*, i.e. a random variable $Z \colon \Omega \to X$ for which $\varphi(t, \omega, Z(\omega)) = Z(\theta_t \omega)$. In particular,

  $$t \mapsto \varphi(t, \cdot, Z)$$

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- One-to-one correspondence between "Markovian" invariant measures for RDS and (classical) invariant measures for Markov semigroups (*Crauel '91*).

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We will ask for existence of invariant measures for RDS induced by RDEs

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An explicit example

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Then

$$Z(\omega) := \begin{cases} \int_{-\infty}^{0} e^{-\alpha s} \,\mathrm{d}X_s(\omega) & \text{for } \alpha < 0 \\ -\int_{0}^{\infty} e^{-\alpha s} \,\mathrm{d}X_s(\omega) & \text{for } \alpha > 0 \end{cases}$$

is a stationary state.

Existence of invariant measures

Theorem.

Let $\mathbf{X}$ be a rough paths cocycle. If b and σ have compact support, the cocycle map induced by the RDE

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admits an invariant measure.

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Proof. Can find a compact subset of $\mathbb{R}^m$ on which cocycle can be defined; existence follows by standard fixed-point theorems (cf. [Crauel, 2002]).

References

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Thank you.