Random dynamical systems and rough paths

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Main tool: *Multiplicative Ergodic Theorem*; yields existence of Lyapunov exponents for linear systems (Oseledets ’68).
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Main tool: *Multiplicative Ergodic Theorem*; yields existence of *Lyapunov exponents* for linear systems (Oseledets ’68).

For non-linear systems, one may look at their linearization, and the MET can still provide information about the local behaviour of the non-linear system (*stable manifold theorem*, Ruelle ’79).
Stable manifold theorem for SDE. (Mohammed-Scheutzow '99)
Assume
\[ dY_t^\xi = b(Y_t^\xi) \, dt + \sigma(Y_t^\xi) \circ dB_t(\omega), \quad Y_0^\xi = \xi \in \mathbb{R}^m \]
induces stochastic (semi-)flow and solution admits an ergodic invariant measure.
Stable manifolds for SDE

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- Differential of the flow solves a linear equation; existence of Lyapunov exponents follow from the MET (under certain assumptions):

\[ \mathbb{P} - \lim_{t \to \infty} \frac{1}{t} \log \left| DY_t^\xi(\omega) v \right| \in \{ \lambda_l < \ldots < \lambda_1 \}, \]

\[ 1 \leq l \leq m, \; v \in \mathbb{R}^m. \]
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  \[ 1 \leq l \leq m, \; v \in \mathbb{R}^m. \]

- Possible to conclude existence of stable (or unstable) manifolds around stationary solutions for the original equation (following Ruelle’s strategy).
Remark.

Itô-theory not really necessary; only used to make sense of initial equation. Markov property used only to speak about invariant measures. In fact, not really necessary, too, more general concept of invariant measures available in the theory of RDS (cf. below).

One of our goals: Build a bridge between the "two cultures" (L. Arnold) Dynamical systems and Stochastic analysis.
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- One of our goals: Build a bridge between the “two cultures” (L. Arnold) Dynamical systems and Stochastic analysis.
**Long term goal:** Prove more general stable manifold theorem for

\[ dY_t = b(Y_t) \, dt + \sigma(Y_t) \, dX_t(\omega) \quad (1) \]

with \( t \mapsto X_t \) rough paths lift of a stochastic process (e.g. fBm).
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Moreover, we will be interested in:

- equilibrium of the system, invariant measures
- long time behaviour (convergence towards equilibrium, attractors, ...)
- qualitative difference between random and deterministic system (stabilization, synchronization by noise, ...)

Remark: In general, solution \( Y \) to (1) not Markovian.
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1 Random dynamical systems: Motivation

2 Random dynamical systems and rough paths

3 Invariant measures for RDEs
Definition.

$(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ called metric dynamical system if

(i) $(\omega, t) \mapsto \theta_t \omega$ is measurable

(ii) $\theta_0 = \text{Id}_\Omega$

(iii) $\theta_{t+s} = \theta_t \circ \theta_s$

(iv) $\mathbb{P} = \mathbb{P} \circ \theta_t^{-1}$ for all $t \in \mathbb{R}$. 

Example: $\Omega = C_0(\mathbb{R}, \mathbb{R}^d)$, Wiener shift: $\theta_t \omega := \omega(t+\cdot) - \omega(t)$. 

If (iv) holds, coordinate process has stationary increments.
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Example: $\Omega = C_0(\mathbb{R}, \mathbb{R}^d)$, $\theta$ Wiener shift:

$$\theta_t \omega := \omega(t + \cdot) - \omega(t).$$

If (iv) holds, coordinate process has **stationary increments**.
Assume that $\bar{X}$ is $\mathbb{R}^d$-valued process defined on some probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ with $\bar{X}(\bar{\omega}) \in C_0(\mathbb{R}, \mathbb{R}^d)$ for all $\bar{\omega} \in \bar{\Omega}$. 
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- If $\bar{X}$ has stationary increments, can build MDS as follows:
Construction of MDS from processes with stationary increments

Assume that $\tilde{X}$ is $\mathbb{R}^d$-valued process defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\tilde{X}(\tilde{\omega}) \in C_0(\mathbb{R}, \mathbb{R}^d)$ for all $\tilde{\omega} \in \tilde{\Omega}$.

- If $\tilde{X}$ has stationary increments, can build MDS as follows:
  - $\Omega := C_0(\mathbb{R}, \mathbb{R}^d)$
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- Coordinate process $X$ on this space has same law as $\tilde{X}$ and satisfies the **cocycle** (or **helix**) property:

\[
X_{t+s}(\omega) - X_s(\omega) = X_t(\theta_s \omega).
\]
Definition.
A \textit{continuous random dynamical system} (RDS) on a topological space \(X\) is a metric DS \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t))\) with mapping

\[ \varphi: [0, \infty) \times \Omega \times X \to X \]

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Definition.
A continuous random dynamical system (RDS) on a topological space $X$ is a metric DS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t))$ with mapping

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(i) $\varphi$ is measurable and $(t, \xi) \mapsto \varphi(t, \omega, \xi)$ is continuous for all $\omega \in \Omega$.

(ii) $\varphi(0, \omega, \cdot) = \text{Id}_X$ for all $\omega \in \Omega$ and

$$\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \cdot) \circ \varphi(s, \omega, \cdot) \quad \text{“cocycle property”}$$

for all $s, t \in [0, \infty)$ and all $\omega \in \Omega$. 

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Example: \( \varphi(t, \omega, \xi) := \phi(0, t, \omega, \xi) \) where

\[
\phi: \left[ 0, \infty \right] \times \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m
\]

solution (semi-)flow to SDE

\[
dY = b(Y) \, dt + \sigma(Y) \circ dB
\]

if \( b, \sigma \) sufficiently “nice”. 

Example: SDEs
Cocycle property for Itō-SDEs

- Using Itō’s theory creates nullsets which depend on all data of the equation.
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In the literature, there are perfection theorems available which can be used to obtain perfect, indistinguishable versions of crude cocycles (Arnold-Scheutzow ’95, Scheutzow ’96). Typical assumption: continuity of crude cocycle.
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In the literature, there are perfection theorems available which can be used to obtain perfect, indistinguishable versions of crude cocycles (Arnold-Scheutzow ’95, Scheutzow ’96). Typical assumption: continuity of crude cocycle.

Not a problem for a pathwise calculus.
Definition.

An MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t))$ together with a rough path valued process $X$ defined on this probability space is called a rough paths cocycle if the cocycle relation

$$X_{s+t}(\omega) = X_s(\omega) \otimes X_t(\theta_s \omega)$$

holds for every $s, t$ and $\omega \in \Omega$. \hfill (2)
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Remark.

- (2) coincides with L. Arnold’s notion of the cocycle property for group-valued processes.
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- on the first level, property (2) reads

$$X_{s+t}(\omega) = X_s(\omega) + X_t(\theta_s \omega).$$
Theorem (Bailleul, R., Scheutzow '17).

If $X$ is a rough paths cocycle, the flow of the rough differential equation

$$dY = b(Y) \, dt + \sigma(Y) \, dX,$$

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Proof.
Straightforward!
More interesting question: Existence of rough cocycles.
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**Theorem (Bailleul, R., Scheutzow).**

If a geometric rough paths valued process $\bar{X}$ has stationary increments, there is a rough paths cocycle $X$ with the same law as $\bar{X}$. 
Existence of rough cocycles

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**Proof.**
As for $\mathbb{R}^d$-valued processes.
Theorem (Bailleul, R., Scheutzow).
If $X$ has stationary increments and the iterated integrals of $X^\varepsilon$ converge towards a rough paths valued process $\bar{X}$ in law, there is a rough paths cocycle $X$ with the same law as $\bar{X}$.

Example. $f$Bm with Hurst parameter $H \in (1/4, 1/2]$. 
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Example.

fBm with Hurst parameter $H \in (1/4, 1/2]$. 
Outline

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2. Random dynamical systems and rough paths
3. Invariant measures for RDEs
Definition.
Let \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t), \varphi)\) be a RDS on a measurable space \((X, \mathcal{B})\). Then the mapping

\[\Theta_t : \Omega \times X \to \Omega \times X\]

\[(\omega, x) \mapsto (\theta_t \omega, \varphi(t, \omega, x))\]

is called skew product. A probability measure \(\mu\) on \(\mathcal{F} \otimes \mathcal{B}\) is called invariant for \(\varphi\) if it has first marginal \(\mathbb{P}\) and if \(\mu \circ \Theta_t^{-1} = \mu\) for all \(t \geq 0\).
Invariant measures II

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- $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}, \mu, (\Theta_t)_{t \geq 0})$ itself MDS.
- Existence of invariant measures implies existence of a stationary state, i.e. a random variable $Z : \Omega \to X$ for which $\varphi(t, \omega, Z(\omega)) = Z(\theta_t \omega)$. In particular,

$$t \mapsto \varphi(t, \cdot, Z)$$

is stationary.
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  is stationary.
- One-to-one correspondence between “Markovian” invariant measures for RDS and (classical) invariant measures for Markov semigroups (Crauel ’91).
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We will ask for existence of invariant measures for RDS induced by RDEs

\[
dY = b(Y) \, dt + \sigma(Y) \, dX(\omega).
\]
Example.
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Then

\[ Z(\omega) := \begin{cases} 
\int_{-\infty}^{0} e^{-\alpha s} \, dX_s(\omega) & \text{for } \alpha < 0 \\
-\int_{0}^{\infty} e^{-\alpha s} \, dX_s(\omega) & \text{for } \alpha > 0
\end{cases} \]

is a stationary state.
Theorem.

Let $X$ be a rough paths cocycle. If $b$ and $\sigma$ have compact support, the cocycle map induced by the RDE

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admits an invariant measure.
**Theorem.**

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**Proof.** Can find a compact subset of $\mathbb{R}^m$ on which cocycle can be defined; existence follows by standard fixed-point theorems (cf. [Crauel, 2002]).
L. Arnold.
Random Dynamical Systems.

Random dynamical systems, rough paths and rough flows.

H. Crauel.
Random Probability Measures on Polish Spaces.

S. Mohammed, M. Scheutzow.
The stable manifold theorem for stochastic differential equations.
Thank you.