On the Hausdorff dimension of a Weierstrass curve whose components are not controlled

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SPDE and multiplication of singular distributions

stochastic reaction-diffusion equations:

$$\partial_t u(t,x) = \Delta u(t,x) + [u(t,x) - u^3(t,x)] + \xi(t,x), \quad t \geq 0, x \in \mathbf{R}^3,$$

Φ_3^4 quantum field:

$$\partial_t \varphi(t,x) = \Delta \varphi(t,x) + \frac{\lambda}{4!}\varphi^3(t,x) + \xi(t,x), \quad t \geq 0, x \in \mathbf{T}^3,$$

$\mathbf{T}^3$: three-d torus, φ^3 renormalized third power of field, ξ space-time noise;

Problem: precise understanding of u^3 or φ^3: *how to multiply singular distributions?*

Method: **paracontrolled distributions;** for renormalization: Hairer's regularity theory.

Multiplication of singular distributions; integration

Pb (singular SPDE): multiplication of singular distributions $f, \dot{g}$ on Euclidean space.

Idea: use Fourier analysis to write $f = \sum_p f_p, \dot{g} = \sum_q \dot{g}_q$ with $f_p, \dot{g}_q$ smooth functions (*Paley-Littlewood blocks*). Then

$$f\dot{g} = \sum_{|p-q|\geq 2} f_p \dot{g}_q + \sum_{|p-q|\leq 1} f_p \dot{g}_q.$$

First term: **para***product*: well behaved, second term: *resonant term*, treated by concepts of **control**. Lit: Gubinelli, I, Perkowski '16, '17.

To be closer to setting of rough path analysis by Lyons, recast problem in the framework of rough integration.

Let $f, g : [0,1] \to \mathbb{R}^d$ be α-Hölder continuous. **Aim:** Define

$$\int f(s) dg(s) = \int f(s) \dot{g}(s) ds.$$

Perspective of stochastic analysis: typically $\alpha < \frac{1}{2}$.

Integration

Let $f, g : [0, 1] \to \mathbb{R}$ be continuous, $f \in \mathcal{C}^{\alpha}, g \in \mathcal{C}^{\beta}$ for $\alpha, \beta \in]0, 1[$. Fourier approach for rough path integral $\int f dg$: expand f, g in (pth) Schauder blocks:

$$f = \sum_{p \geq 0} \Delta_p f, \quad g = \sum_{q \geq 0} \Delta_q g,$$

$$\Delta_p h = \sum_{m=1}^{2^p} \langle H_{pm}, dh \rangle G_{pm}.$$

Haar functions, Schauder functions: $l_{pm} = \frac{m-1}{2^p}, c_{pm} = \frac{2m-1}{2^{p+1}}, u_{pm} = \frac{m}{2^p}$;

$$H_{pm} = 2^{p/2}[1_{[l_{pm}, c_{pm}[} - 1_{[c_{pm}, u_{pm}[}], \quad G_{pm} = \int_0^{\cdot} H_{pm}(s) ds.$$

Then define the integral by

$$\int f dg = \sum_{p,q} \int \Delta_p f d\Delta_q g.$$

Decomposition of the integral

Decompose the (formal) integral into three components with essentially different smoothness properties. We may write with $S_p f = \sum_{q \le p} \Delta_q f$

$$\begin{aligned}\int f dg &= \sum_{p,q} \int \Delta_p f d\Delta_q g \\ &= \sum_{p<q} \int \Delta_p f d\Delta_q g + \sum_{p \ge q} \int \Delta_p f d\Delta_q g \\ &= \sum_q \int S_{q-1} f d\Delta_q g + \sum_p \int \Delta_p f d\Delta_p g + \sum_p \int \Delta_p f dS_{p-1} g.\end{aligned}$$

In view of the second part of Corollary 1, we expect the first part to be rougher. Integration by parts gives

$$\begin{aligned}\sum_q \int S_{q-1} f d\Delta_q g &= \sum_q S_{q-1} f \Delta_q g - \sum_q \int \Delta_q g dS_{q-1} f \\ &= \pi_<(f,g) - \sum_q \int \Delta_q g dS_{q-1} f.\end{aligned}$$

Decomposition of the integral

$\pi_<(f,g)$: *Bony paraproduct*

Defining further

$$L(f,g) = \sum_p (\Delta_p f dS_{p-1} g - \Delta_p g dS_{p-1} f),$$

(*antisymmetric Lévy area*)

$$S(f,g) = \sum_p \Delta_p f d\Delta_p g = c + \frac{1}{2} \sum_p \Delta_p f \Delta_p g$$

(*symmetric part*)

we have

$$\int f dg = \pi_<(f,g) + S(f,g) + L(f,g).$$

The Young integral

In case the Hölder regularity coefficients of f and g are large enough, the three components of the integral possess the following regularity.

For any $\alpha, \beta \in]0,1[$

$$||S(f,g)||_{\alpha+\beta} \leq C||f||_{\alpha}||g||_{\beta}.$$

Similarly

$$||\pi_{<}(f,g)||_{\beta} \leq C||f||_{\infty}||g||_{\beta}.$$

and, **but only if** $\alpha + \beta > 1$

$$||L(f,g)||_{\alpha+\beta} \leq C||f||_{\alpha}||g||_{\beta}.$$

This provides **Young's integral if** $\alpha + \beta > 1.$

Problem: What happens if $\alpha + \beta \leq 1$ (the rough path case)?
(Para)controlledness of integrand w.r.t. integrator needed, i.e. increments of integrand can be developed into fractional Taylor expansion in increments of integrator.

Beyond Young's integral: Weierstrass curve

Pb: geometric meaning of existence of integral; consider **Weierstrass functions**

$$W_1(x) := \sum_{k=1}^{\infty} a_k \cos(2^k \pi x), \quad W_2(x) := \sum_{k=1}^{\infty} a_k \sin(2^k \pi x), \quad a_k := 2^{-\alpha k}, \alpha \in]0,1[.$$

By summability of a_k: Weierstrass functions are uniform limits of partial sums. For $x, y \in [-1, 1]$ and $k \in \mathbb{N}$ such that $2^{-k} \leq |x-y| \leq 2^{-k+1}$:

$$\begin{aligned} |W_1(x) - W_1(y)| &\leq C\,[\sum_{l=1}^{k} a_l 2^l |x-y| + \sum_{l=k+1}^{\infty} a_l] \\ &\leq C[2^{k(1-\alpha)}|x-y| + 2^{-k\alpha}] \leq C|x-y|^{\alpha}. \end{aligned}$$

Hence W_1, W_2 α-Hölder continuous.

Simple arguments: for $2\alpha \leq 1$: W_1, W_2 **mutually not controlled,**
$W = (W_1, W_2)$ **has no Lévy area!**

Weierstrass curve: no control

Take $x = 0, y = 2^{-n}$, $V(0) > 0$ ($V(0) < 0$ analogous):

$$
\begin{aligned}
&|W_2(y) - W_2(0) - V(0)(W_1(y) - W_1(0))| \\
&\quad = \left| \sum_{k=1}^{\infty} a_k [(\sin(2^k \pi y) - V(0)(\cos(2^k \pi y) - 1)] \right| \\
&\quad = \left| 2 \sum_{k=1}^{\infty} a_k \left[\sin(2^k \pi \frac{y}{2}) \cos(2^k \pi \frac{y}{2}) + V(0) \sin(2^k \pi \frac{y}{2}) \sin(2^k \pi \frac{y}{2}) \right] \right| \\
&\quad = \left| 2 \sum_{k=1}^{\infty} a_k \sin(2^k \pi \frac{y}{2}) \sqrt{1 + V(0)^2} \sin[2^k \pi \frac{y}{2} + \arctan((V(0))^{-1})] \right| \\
&\quad = \left| 2 \sum_{k=1}^{n} a_k \sin(2^{k-1-n} \pi) \sqrt{1 + (V(0))^2} \sin[2^{k-1-n} \pi + \arctan((V(0))^{-1})] \right| \\
&\quad \geq 2^{-\alpha n} \sin\left(\frac{\pi}{2} + \arctan((V(0))^{-1})\right) \\
&\quad \neq \mathcal{O}(|y - x|^{2\alpha}).
\end{aligned}
$$

Hence W_1 is not controlled by W_2, and vice versa by analogy.

Weierstrass curve: no Lévy area

$$L(W_1^m, W_2^m) = \int_{-1}^{1} W_1^m(x) dW_2^m(x) - \int_{-1}^{1} W_2^m(x) dW_1^m(x)$$

$$= \sum_{k,l=1}^{m} a_k a_l \int_{-1}^{1} \big(\sin(2^k \pi x) \sin(2^l \pi x) 2^l \pi + \cos(2^l \pi x) \cos(2^k \pi x) 2^k \pi \big) dx$$

$$= \sum_{k,l=1}^{m} a_k a_l \big(2^l \pi \int_{-1}^{1} \frac{1}{2} (\cos((2^k - 2^l)\pi x) - \cos((2^k + 2^l)\pi x)) dx$$

$$+ 2^k \pi \int_{-1}^{1} (\cos((2^k - 2^l)\pi x) + \cos((2^k + 2^l)\pi x)) dx \big)$$

$$= 2 \sum_{k=1}^{m} a_k^2 2^k \pi = 2 \sum_{k=1}^{m} 2^{(1-2\alpha)k} \pi.$$

This diverges as m tends to infinity for $\alpha \leq \frac{1}{2}$. Hence $W = (W_1, W_2)$ possesses no Lévy area.

The geometry of the Weierstrass curve

Here is a plot of W for $\alpha = \frac{1}{2}$:

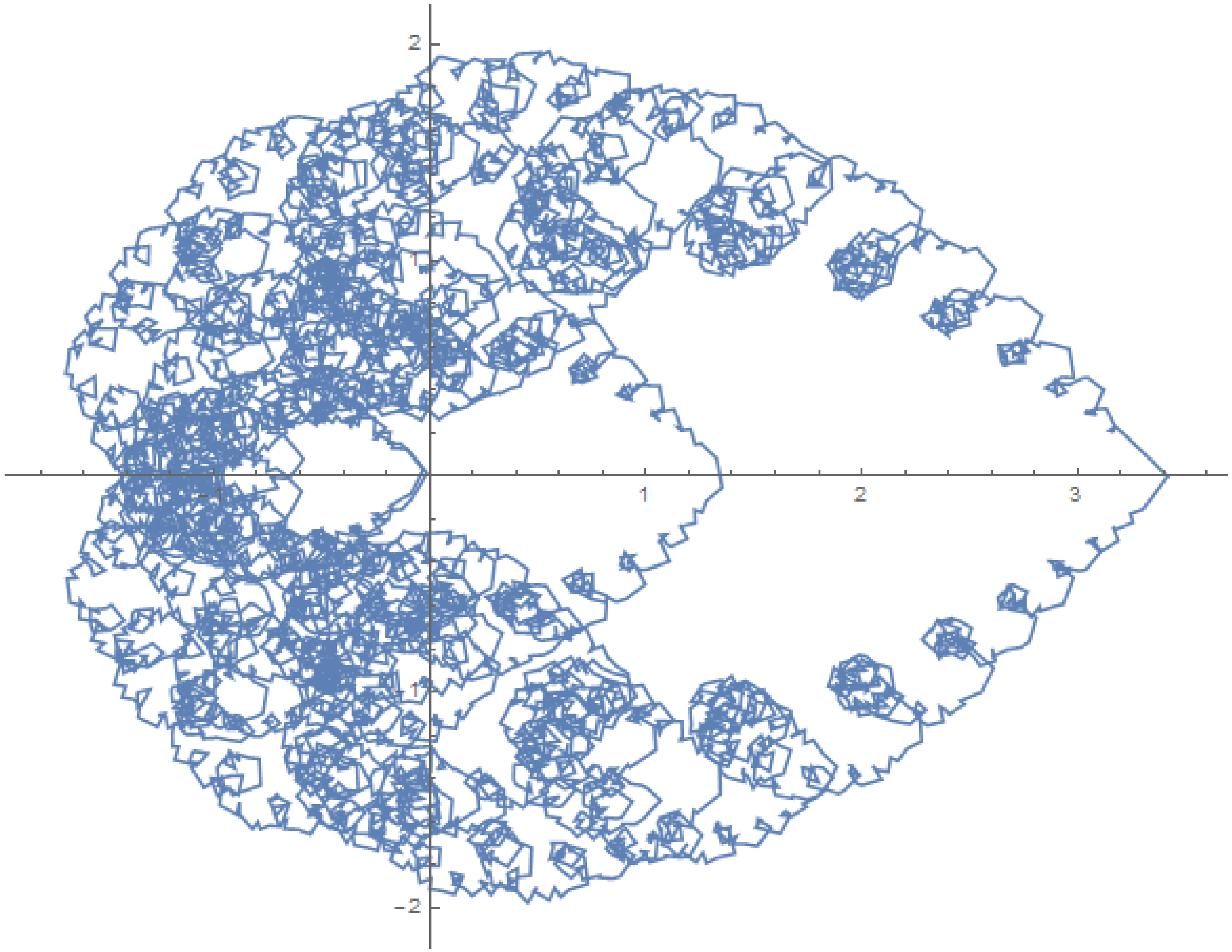

Abbildung 1: Parametric plot of W for $x \in [0,1]$; $\{W(x) : x \in [0,1]\} \subset \mathbb{R}^2$.

A metric dynamical system

Lit: Baranski et al. '14, Keller '15, Shen '15.

Goal: Let $\alpha = \frac{1}{2}$. Describe W as attractor of a dynamical system on $[0,1]^2$, alternatively $\Omega = \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

For $\omega \in \Omega$, write $\omega = ((\omega_{-n})_{n\geq 0}, (\omega_n)_{n\geq 1})$, **F** product σ-field.

Canonical shift on Ω:

$$\theta : \Omega \to \omega, \omega \mapsto (\omega_{n+1})_{n\in\mathbb{Z}}, \quad \nu = \otimes_{n\in\mathbb{Z}}(\frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}})$$

the infinite product of Bernoulli measures.

$(\Omega, \mathbf{F}, \nu, \theta)$ metric dynamical system.

A metric dynamical system, Baker's transformation

Now let

$$T = (T_1, T_2) : \Omega \to [0,1]^2, \quad \omega \mapsto (\sum_{n=0}^{\infty} \omega_{-n} 2^{-(n+1)}, \sum_{n=1}^{\infty} \omega_n 2^{-n}).$$

Then $\nu = \lambda^2 \circ T$, λ Lebesgue measure on $[0,1]^2$. T^{-1}: dyadic representation of components in $[0,1]^2$. Let

$$B = T \circ \theta \circ T^{-1} \qquad \text{Baker's transformation.}$$

The ν-invariance of θ implies B-invariance of λ^2. For $(\xi, x) \in [0,1]^2$ denote

$$T^{-1}(\xi, x) = \left((\overline{\xi}_{-n})_{n\geq 0}, (\overline{x}_n)_{n\geq 1}\right).$$

For $(\xi, x) \in [0,1]^2$ and $k \geq 0$ resp. $k \geq 1$

$$B(\xi, x) = \left(2\xi, \frac{\overline{\xi}_0 + x}{2}\right), \quad B^{-1}(\xi, x) = \left(\frac{\xi + \overline{x}_1}{2}, 2x\right).$$

Self affinity: W as attractor of a random dynamical system

Extend W from $[0,1]$ to $[0,1]^2$ by $W(\xi,x) = W(x), \quad \xi, x \in [0,1]$.

By 2π-periodicity of trigonometric functions

$$\begin{aligned}
W\big(B_2(\xi,x)\big) &= W\Big(\frac{\overline{\xi}_0+x}{2}\Big) = \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} \left(\begin{array}{c}\cos\\ \sin\end{array}\right) \Big(2\pi 2^n \frac{\overline{\xi}_0+x}{2}\Big) \\
&= \left(\begin{array}{c}\cos\\ \sin\end{array}\right) \Big(2\pi \frac{\overline{\xi}_0+x}{2}\Big) + \sum_{n=1}^{\infty} 2^{-\frac{n}{2}} \left(\begin{array}{c}\cos\\ \sin\end{array}\right) (2\pi 2^{n-1} x) \\
&= \left(\begin{array}{c}\cos\\ \sin\end{array}\right) \Big(2\pi \frac{\overline{\xi}_0+x}{2}\Big) + 2^{-\frac{1}{2}} \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} \left(\begin{array}{c}\cos\\ \sin\end{array}\right) (2\pi 2^{n} x) \\
&= \left(\begin{array}{c}\cos\\ \sin\end{array}\right) \big(2\pi B_2(\xi,x)\big) + 2^{-\frac{1}{2}} W(\xi,x).
\end{aligned}$$

W as attractor of a random dynamical system

Define the map

$$\begin{aligned} F : [0,1]^2 \times \mathbb{R}^2 &\to [0,1]^2 \times \mathbb{R}^2, \\ (\xi, x, y_1, y_2) &\mapsto \Big(B(\xi,x), 2^{-\frac{1}{2}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \cos \\ \sin \end{pmatrix} (2\pi B_2(\xi,x))\Big), \end{aligned}$$

where $B = (B_1, B_2)$.

Then

$$\Big(B(\xi,x), W(B(\xi,x))\Big) = \Big(B(\xi,x), W(B_2(\xi,x))\Big) = F\Big(\xi, x, W(\xi,x)\Big).$$

Hence W is an attractor for F.

Lyapunov exponents and invariant structures

Calculate Jacobian: for $\xi, x \in [0,1], y_1, y_2 \in \mathbb{R}$

$$DF(\xi, x, y_1, y_2) \quad = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\pi \sin\big(2\pi B_2(\xi, x)\big) & 2^{-\frac{1}{2}} & 0 \\ 0 & \pi \cos\big(2\pi B_2(\xi, x)\big) & 0 & 2^{-\frac{1}{2}} \end{bmatrix}.$$

Hence Lyapunov exponents of F: $2, \frac{1}{2}, \gamma := 2^{-\frac{1}{2}}$, the last being double.

Invariant vector fields: if $S(\xi, x) = -2\pi \sum_{n=1}^{\infty} \gamma^n \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \big(2\pi B_2^n(\xi, x)\big)$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X(\xi, x) = \begin{pmatrix} 0 \\ 1 \\ S(\xi, x) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence X spans invariant stable manifold: for $\xi, x \in [0,1], y_1, y_2 \in \mathbb{R}$

$$DF(\xi, x, y_1, y_2) X(\xi, x) = X\big(B(\xi, x)\big).$$

The Sinai-Bowen-Ruelle measure

Idea: Calculate Hausdorff dimension of graph of W by approach of Ledrappier-Young for calculation of Hausdorff dimension of attractors.

Calculate action of S on λ^2-measure preserving map B: for $\xi, x \in [0,1]$

$$\begin{aligned}
S(B(\xi,x)) &= -2\pi \sum_{n=1}^{\infty} \gamma^n \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \left(2\pi B_2^n(B_2(\xi,x))\right) \\
&= -2\pi \sum_{n=1}^{\infty} \gamma^n \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \left(2\pi B_2^{n+1}(\xi,x)\right) \\
&= -2\pi 2^{\frac{1}{2}} \sum_{k=1}^{\infty} \gamma^k \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \left(2\pi B_2^k(\xi,x)\right) + 2\pi \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \left(2\pi B_2(\xi,x)\right) \\
&= 2^{\frac{1}{2}} S(\xi,x) + 2\pi \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \left(2\pi B_2(\xi,x)\right).
\end{aligned}$$

The Sinai-Bowen-Ruelle measure

So we may define

$$G : [0,1]^2 \times \mathbb{R}^2 \to [0,1]^2 \times \mathbb{R}^2,$$
$$(\xi, x, v_1, v_2) \mapsto \Big(B(\xi,x), 2^{\frac{1}{2}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + 2\pi \begin{pmatrix} -\sin \\ \cos \end{pmatrix} (2\pi B_2(\xi,x))\Big).$$

And for $\xi, x \in [0,1]$ we have

$$G\big(\xi, x, S(\xi,x)\big) = \big(B(\xi,x), S(B(\xi,x))\big).$$

The measure

$$\psi = \lambda^2 \circ (\mathsf{id}, S)^{-1}$$

on $\mathcal{B}([0,1]^2) \otimes \mathcal{B}(\mathbb{R}^2)$ is G-invariant. Define $\pi_2 : [0,1]^2 \to [0,1], (\xi,x) \mapsto x$ and let

$$\mu = \lambda^2 \circ (\pi_2, S)^{-1}.$$

The measure μ is the *Sinai-Bowen-Ruelle measure* of G.

The absolute continuity of the SBR measure

Let $K_r(z) = \{y \in \mathbb{R}^2 : |y - z| < r\}$ for $r > 0, z \in \mathbb{R}^2$. For Borel measures ρ, ρ' on $\mathcal{B}(\mathbb{R}^2), r > 0$ let $\langle \rho, \rho' \rangle_r = \int_{\mathbb{R}^2} \rho\big(K_r(z)\big)\rho'\big(K_r(z)\big)dz$ and $\|\rho\|_r = \langle \rho, \rho \rangle_r^{\frac{1}{2}}$. Then if ρ is probability measure on $\mathcal{B}(\mathbb{R}^2)$, and $\limsup_{r\to 0} \frac{1}{r^2}||\rho||_r < \infty$, $\rho << \lambda^2$.

Lemma 1. *For $r > 0$ let*

$$I(r) = \frac{1}{r^4} \int_0^1 ||\mu_x||_r^2 dx.$$

If $\limsup_{r\to 0+} I(r) < \infty$, then μ is absolutely continuous w.r.t. Lebesgue measure with square integrable density.

With this criterion we show

Theorem 1. *We have*

$$\limsup_{r\to 0+} I(r) < \infty.$$

The SBR measure μ is absolutely continuous with square integrable density.

The Hausdorff dimension of W

Upper bound: Elementary arguments using coverings prove that Hausdorff diimension of graph of W is bounded above by 2.

Lower bound: Need to show that there is a measure supported by the graph, calculate its local dimension. Use the measure

$$m = \lambda^2 \circ (\mathrm{id}, W)^{-1}.$$

Calculate lower bound for *local dimension* of m on the graph.

For $K > 0, N \in \mathbb{N}, \xi, x \in [0,1]$ let $I_N(x)$ be neighborhood of x of diameter 2^{-N}, and

$$V_N(\xi, x) = \left\{(r, w) \in [0,1] \times \mathbb{R}^2 : r \in I_N(x), |w - l_{(\xi,x,W(x))}(r)| \leq K \cdot 2^{-N}\right\},$$

where $l_{(\xi,x,v)}$ is the motion through $(\xi, x, W(\xi, x))$ along the stable fiber described by $S(\xi, x)$ given by solutions of

$$\frac{d}{dr} l_{(\xi,x,v)}(r) = S(\xi, r), \quad \text{with} \quad l_{(\xi,x,v)}(x) = v, \quad v \in \mathbb{R}^2.$$

The Hausdorff dimension of W

Calculate lower bound for

$$\liminf_{N\to\infty} \frac{\log m\big(V_N(\xi,x)\big)}{\log 2^{-N}}.$$

By a scaling argument, for $N \in \mathbb{N}$

$$\frac{\log m(V_N(\xi,x))}{\log 2^{-N}}$$
$$= 1 + \frac{\log \lambda\big(\{u \in [0,1] : |W(u) - l_{(B^{-N}(\xi,x)),W(B^{-N}(\xi,x)))}(u)| \leq K\gamma^N\}\big)}{\log 2^{-N}}.$$

The latter quantity is computed along a technical telescoping argument by Keller using: geometry of stable manifolds near attractor, absolute continuity of SBR measure, and a local time of $|W|$ at 0.

The Hausdorff dimension of W

This leads to

Proposition 1. *We have*

$$\liminf_{N\to\infty} \frac{\log \lambda(\{u \in [0,1] : |W(u) - l_{(B^{-N}(\xi,x),W(B^{-N}(\xi,x)))}(u)| \le K\gamma^N\})}{\log 2^{-N}} \ge 1.$$

This implies our final result.

Theorem 2. *Let* $m = \lambda^2 \circ (id, W)^{-1}$, $x \in [0,1]$. *Then*

$$\liminf_{N\to\infty} \frac{\log m(V_N(\xi,x))}{\log 2^{-N}} \ge 2.$$

The Hausdorff dimension of the graph of W is 2.