On Constrained Pathwise Stochastic Differential Equations

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Introduction

The model

$$X(t) = X_0 + \int_0^t b(X(s))\mathrm{d}s + \int_0^t \sigma(X(s))\mathrm{d}B(s) \tag{1}$$

where B is a fractional Brownian motion.

Sense of the stochastic integral specified later.

How to constrain X to stay in a convex subset of $\mathbb{R}^e$?

If B is a Brownian motion, in Itô's calculus framework:

- Invariance condition on (b, σ).
- Skorokhod reflection problem.

Objectives

If B is a fractional Brownian motion, to extend the previous methods in the pathwise stochastic calculus framework.

References: invariance condition

With

$$H \in \left[\frac{1}{2}, 1\right[$$

in the fractional calculus framework:

- Ciotir & Rascanu (2009)
- Nie & Rascanu (2011)
- Melnikov & al. (2015)

In the rough paths framework:

- Coutin & NM (2017) $\leftarrow$

References: reflection in the rough paths framework

Existence with a constant constraint set:

- Aida (2015,2016)

Existence and uniqueness with a constant constraint set:

- Besalu & al. (2012): nonnegative constraints.
- Deya & al. (2016): one-dimensional constraint set.

Existence and uniqueness with time-dependent constraint sets:

- Falkowski & Slominski (2015):
  $H \in \left] \frac{1}{2}, 1 \right[$ and time-dependent cuboïd contraints.
- Castaing, NM and Raynaud de Fitte (2017) ←

Invariance theorem for rough differential equations

Invariant sets

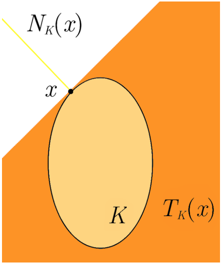
A subset K of $\mathbb{R}^e$ is invariant by (1) if and only if

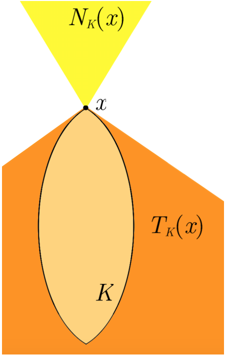
$$\forall X_0 \in K, \forall X \text{ solution of (1)}, X(\Omega \times [0,T]) \subset K.$$

Tangent and normal cones to a closed convex set K at x

$$N_K(x) := \{s : \forall y \in K,\ \langle s, y - x\rangle \leqslant 0\}$$

$$T_K(x) := \{\delta : \forall s \in N_K(x),\ \langle s, \delta\rangle \leqslant 0\}$$





Rough invariance theorem

Coutin & NM (2017)

If (1) has solutions, then K is invariant by (1) if and only if

$\mathfrak{C}(\partial K)$: $\forall x \in \partial K, \forall k \in [\![1, e]\!], b(x), \pm\sigma_{.,k}(x) \in T_K(x).$

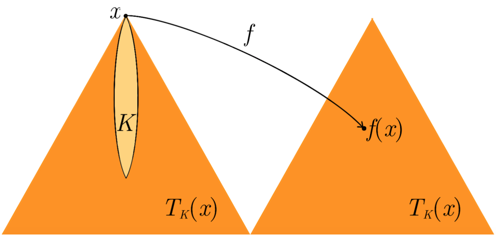


Figure: $f = b$; $\pm\sigma_{.,k}$

Example: K is a vector subspace

Since

$$\partial K = K$$

and

$$T_K(x) = K \; ; \forall x \in \partial K,$$

$\mathfrak{C}(\partial K)$ is equivalent to

$$b(K) \subset K \text{ and } \sigma_{.,k}(K) \subset K \tag{2}$$

Example: K is a half-space

Since

$$\partial K = \{x : \langle v, x - a\rangle = 0\}$$

and

$$T_K(x) = \{\delta : \langle \delta, v\rangle \leqslant 0\} \ ; \forall x \in \partial K,$$

$\mathfrak{C}(\partial K)$ is equivalent to

$$\langle v, b(x)\rangle \leqslant 0 \text{ and } \langle v, \sigma_{.,k}(x)\rangle = 0 \ ; \forall x \in \partial K \qquad (3)$$

Example: K is the unit ball

Since

$$\partial K = \{x : \|x\| = 1\}$$

and

$$T_K(x) = \{\delta : \langle \delta, x \rangle \leqslant 0\} \ ; \forall x \in \partial K,$$

$\mathfrak{C}(\partial K)$ is equivalent to

$$\langle x, b(x) \rangle \leqslant 0 \text{ and } \langle x, \sigma_{.,k}(x) \rangle = 0 \ ; \forall x \in \partial K \qquad (4)$$

Sufficient condition: sketch of proof

Consider

$$X_n(t) = X_0 + \int_0^t b(X_n(s))\mathrm{d}s + \int_0^t \sigma(X_n(s))\mathrm{d}B_n(s) \qquad (5)$$

Steps: K is invariant by

1. (5) under $\mathfrak{C}(\mathbb{R}^e)$ by proving that

$$\lim_{h\to 0^+, u\to 1} \frac{d_K(X_n(t+hu))^2 - d_K(X_n(t))^2}{h} \leqslant 0.$$

Sufficient condition: sketch of proof

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2. (5) under $\mathfrak{C}(\partial K)$ by replacing (b,σ) by $(b,\sigma)\circ p_K^{\perp}$ in (5).

Sufficient condition: sketch of proof

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1. (5) under $\mathfrak{C}(\mathbb{R}^e)$ by proving that
$$\lim_{h\to 0^+, u\to 1} \frac{d_K(X_n(t+hu))^2 - d_K(X_n(t))^2}{h} \leqslant 0.$$
2. (5) under $\mathfrak{C}(\partial K)$ by replacing (b,σ) by $(b,\sigma)\circ p_K^{\perp}$ in (5).
3. (1) under $\mathfrak{C}(\partial K)$ because K is closed.

Necessary condition: sketch of proof

Steps: If K is invariant by (1), then $\mathfrak{C}(\partial K)$ is true by considering successively

1. a hyperplane K,
2. a half-space K,
3. a closed and convex set K.

Skorokhod reflection problem with time-dependent constraint sets

The unperturbed sweeping process

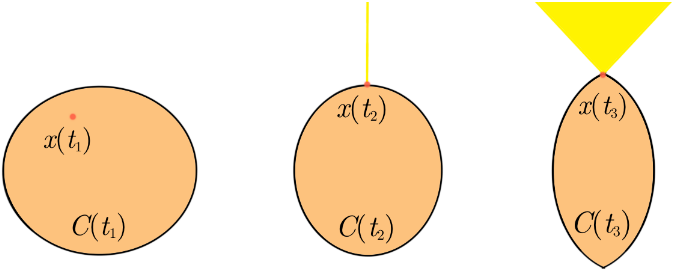
Defined by the differential inclusion

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) \text{ a.e.} \\ x(0) = x_0 \in C(0) \end{cases} \tag{6}$$

where $C : [0, T] \rightrightarrows \mathbb{R}^e$ is a convex compact valued multifunction, continuous for the Hausdorff distance.

The unperturbed sweeping process: some pictures

Three situations:



The Monteiro-Marques theorem

If there exist $a \in \mathbb{R}^e$ and $r > 0$ such that

$$\overline{B}_e(a,r) \subset C(t) \ ; \forall t \in [0,T],$$

then (6) has a unique continuous solution of finite 1-variation.

The rough perturbed sweeping process

Defined by

$$X(t) = \underbrace{\int_0^t b(X(u))\mathrm{d}u + \int_0^t \sigma(X(u))\mathrm{d}B(u)}_{H(t)} + Y(t) \qquad (7)$$

where

$$\begin{cases} -\dot{Y}(t) \in N_{C(t)-H(t)}(Y(t)) \text{ a.e.} \\ Y(0) = X_0 \end{cases} \qquad (8)$$

A continuity theorem
(Castaing et al. (2014))

Consider a continuous function $h : [0, T] \to \mathbb{R}^e$ and

$$\begin{cases} v_h(t) = h(t) + w_h(t) \\ -\dot{w}_h(t) \in N_{C(t)-h(t)}(w_h(t)) \text{ a.e.} \\ w_h(0) = x_0 \in C(0) \end{cases} \quad (9)$$

For every sequence of continuous functions $(h_n)_{n\in\mathbb{N}}$, if

$$h_n \to h \text{ uniformly,}$$

then

$$(v_{h_n}, w_{h_n}) \to (v_h, w_h) \text{ uniformly.}$$

Existence of solutions to (7)-(8)
(Castaing, NM and Raynaud de Fitte (2017))

If there exists a continuous function $\gamma : [0,T] \to \mathbb{R}^e$ such that

$$\overline{B}_e(\gamma(t), r) \subset C(t) \ ; \forall t \in [0,T],$$

then (7)-(8) has at least one solution of finite p-variation with

$$p \in \left] \frac{1}{H}, \infty \right[.$$

Existence of solutions to (7)-(8): sketch of proof

Consider the Picard scheme:

$$\begin{cases} X_n(t) = H_n(t) + Y_n(t) \\ H_n(t) = \int_0^t b\left(X_{n-1}(u)\right) \mathrm{d}u + \int_0^t \sigma\left(X_{n-1}(u)\right) \mathrm{d}B(u) \\ -\dot{Y}_n(t) \in N_{C(t)-H_n(t)}\left(Y_n(t)\right) \text{ a.e. with } Y_n(0) = X_0 \end{cases} .$$

The continuity theorem allows to prove that $(X_n, Y_n)_{n \in \mathbb{N}}$ has a converging subsequence.

Its limit is a solution to (7)-(8).

Uniqueness of the solution to (7)-(8): preliminary

Consider the smooth perturbed sweeping process

$$\begin{cases} x(t) = h(t) + y(t) \\ h(t) = \int_0^t f(x(u))\mathrm{d}u \\ -\dot{y}(t) \in N_{C(t)-h(t)}(y(t)) \text{ a.e. with } y(0) = X_0 \end{cases} \tag{10}$$

Uniqueness of the solution to (7)-(8): preliminary

Consider two solutions (x, y) and (x^*, y^*) to (10).

Since $z \mapsto N_{C(\tau)}(z)$ is monotone,

$$\|x - x^*\|_{\infty,\tau}^2 \leqslant c \int_0^\tau \langle x(u) - x^*(u), \mathrm{d}(y - y^*)(u) \rangle \leqslant 0$$

with $c, \tau > 0$.

Uniqueness of the solution to (7)-(8): sufficient condition
(Castaing, NM and Raynaud de Fitte (2017))

Consider two solutions (X, Y) and (X^*, Y^*) to (7)-(8).

The monotonicity of the normal cone is not sufficient to prove that

$$\|X - X^*\|_{p\text{-var},T} = 0.$$

It is true if

$$\int_s^t \langle R_X(s,u) - R_{X^*}(s,u), \mathrm{d}(Y - Y^*)(u)\rangle \leqslant 0 \qquad (11)$$

Uniqueness of the solution to (7)-(8)
(Castaing, NM and Raynaud de Fitte (2017))

Assume that for any solution (X, Y) to (7)-(8),

$$\mathrm{Int}\{t \in [0, T] : X(t) \in \partial C(t)\} = \emptyset.$$

Then (7)-(8) has a unique solution.

For instance, if C is a time-dependent convex polyhedron and σ is constant, then

$$\mathrm{Int}\{t \in [0, T] : X(t) \in \partial C(t)\} = \emptyset.$$

Thank you for your attention !