Arakelov invariants and the Belyi degree

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October 22, 2012

These notes grew out of talks given at the university of Hamburg on May 16th 2012, the university of Besancon on September 27th 2012, and the REGA seminar at the IHP in Paris on October 10th 2012. The main reference for the statements below is [J].

We permit ourselves to occasionally be slightly informal for the sake of exposition.

1 Introduction

Let K be a number field and let X be a (smooth projective geometrically connected) curve over K.

The aim of this talk is to show that each **Arakelov invariant** of X is bounded by a polynomial in the **Belyi degree** of X.

This is the main result of [J].

To motivate the reader, we mention three applications:

1. The Couveignes-Edixhoven-Bruin algorithm for "computing coefficients of modular forms" runs in **polynomial** time under GRH. This was known for certain congruence subgroups before; see Theorem 5.0.1 in [J] for a precise statement.
2. The Edixhoven-de Jong-Schepers conjecture on the Faltings height of a cover of $\mathbf{P}^1_{\mathbf{Z}}$ is true. We will be more precise later in this talk; see Section 5.
3. Szpiro's small points conjecture holds for curves which are cyclic covers of $\mathbf{P}^1$. We make a precise statement in Section 6.

2 The Belyi degree

Theorem 2.0.1. *Let X be a smooth projective connected curve over $\mathbf{C}$. The following assertions are equivalent.*

1. *The curve X can be defined over a number field.*
2. *There exists a finite morphism $X \to \mathbf{P}^1_{\mathbf{C}}$ ramified over precisely three points. (We call such a morphism a Belyi map.)*

Belyi proved that (1) implies (2). Grothendieck (and in fact Weil) already proved that (2) implies (1).

Example 2.0.2. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a finite index subgroup. Then the compactification X_Γ of the Riemann surface $\Gamma\backslash\mathbf{H}$ (obtained by adding cusps) can be defined over a number field. This follows from the implication (2) $\implies$ (1). In fact, the morphism $X_\Gamma \to X(1) \cong \mathbf{P}^1_{\mathbf{C}}$ of degree $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma]$ is ramified over precisely three points. (The isomorphism $X(1) \cong \mathbf{P}^1(\mathbf{C})$ is given by the j-invariant.)

Example 2.0.3. Let $F(n)$ be the curve $x^n + y^n = z^n$ in $\mathbf{P}^2_{\overline{\mathbf{Q}}}$. Then the morphism $F(n) \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ defined as $(x : y : z) \mapsto (x^n : z^n)$ is ramified over precisely three points. We note that this finite morphism is of degree n^2.

Definition 2.0.4. Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$. Then the Belyi degree of X, denoted by $\deg_{\mathrm{Belyi}}(X)$, is defined as the minimal degree of a finite morphism $X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ ramified over precisely three points. (This is well-defined by the above theorem.)

The Belyi degree is very hard to compute in practice. Nevertheless, it is easy to bound the Belyi degree in general in practice.

Example 2.0.5. The Belyi degree of the curve X_Γ is bounded by the index of Γ in $\mathrm{SL}_2(\mathbf{Z})$.

Example 2.0.6. The Belyi degree of the Fermat curve $F(n)$ is bounded by n^2.

The Belyi degree of a curve has the following remarkable "Northcott" property.

Proposition 2.0.7. *Let C be a real number. The set of $\overline{\mathbf{Q}}$-isomorphism classes of smooth projective connected curves X such that $\deg_{Belyi}(X) \leq C$ is finite.*

Proof. The proof is based on a purely topological argument. In fact, it suffices to note that the fundamental group of the Riemann sphere minus three points is finitely generated. The proof follows from a standard argument involving Galois theory. $\square$

3 Arakelov geometry

Let X be a smooth projective geometrically connected curve over a number field K.

We can do two things.

1. We can consider the arithmetic geometry of X over the ring of integers O_K, i.e., study the arithmetic surfaces attached to X.

2. We can consider the analytic geometry of X over the complex numbers for each embedding $K \to \mathbf{C}$, i.e., study the Riemann surfaces attached to X.

Roughly speaking, Arakelov geometry does these two geometries **simultaneously**.

One can use Arakelov geometry to define Arakelov invariants of X. We mention three of these.

1. The (absolute stable) Faltings height $h_{\mathrm{Fal}}(X)$ of X. This invariant plays a key role in Faltings proof of the Mordell conjecture (1983).

2. The discriminant $\Delta(X)$ of X. This invariant measures in some sense the bad reduction of the curve; see below for a precise definition.

3. The self-intersection of the relative dualizing sheaf $e(X)$ of X. This invariant is related to the Bogomolov conjecture for curves (Szpiro).

Let us briefly explain how $\Delta(X)$ measures the bad reduction.

Let L/K be a finite field extension such that X_L has semi-stable reduction. Let $p : \mathcal{X} \to \operatorname{Spec} O_L$ be the semi-stable minimal regular model of X_L. The discriminant of X is defined as

$$\Delta(X) = \frac{\sum_{\mathfrak{p} \subset O_L} \delta_{\mathfrak{p}} \log \# k(\mathfrak{p})}{[L : \mathbf{Q}]}.$$

Here the sum runs over the maximal ideals $\mathfrak{p}$ of O_L and $\delta_{\mathfrak{p}}$ is the number of singularities in the geometric fibre of p over $\mathfrak{p}$. Note that $\Delta(X) \geq 0$ and that $\Delta(X) = 0$ if and only if X has potentially good reduction over O_K.

We omit the definition of $e(X)$ and $h_{\mathrm{Fal}}(X)$ for the sake of brevity. We don't need them.

4 Main result

Definition 4.0.8. Let X be a smooth projective geometrically connected curve over a number field K. The Belyi degree of X, denoted by $\deg_B(X)$, is the Belyi degree of $X_{\overline{\mathbf{Q}}}$ for some $K \subset \overline{\mathbf{Q}}$.

It is easy to show that $g \leq \deg_B(X)$ using the Riemann-Hurwitz theorem, where g is the genus of X.

We now state our main theorem.

Theorem 4.0.9. *Let X be a smooth projective geometrically connected curve over a number field K of genus g. Then*

$$h_{\mathrm{Fal}}(X) \leq 13 \cdot 10^6 g \deg_B(X)^5,$$
$$\Delta(X) \leq 5 \cdot 10^8 g^2 \deg_B(X)^5,$$
$$e(X) \leq 3 \cdot 10^7 (g-1) \deg_B(X)^5.$$

Remark 4.0.10. Note that the lefthandside is an Arakelov invariant and that the righthandside is a polynomial in the Belyi degree.

5 Application: The Edixhoven-de Jong-Schepers conjecture

We were first led to investigate this problem by a conjecture of Edixhoven, de Jong and Schepers [EdJoSc, Conjecture 5.1]. The following theorem implies this conjecture.

Theorem 5.0.11. *Let $B \subset \mathbf{P}^1(\overline{\mathbf{Q}})$ be a finite set with complement U in $\mathbf{P}^1_{\overline{\mathbf{Q}}}$. Then, for any finite morphism $\pi : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ with X a smooth projective connected curve over $\overline{\mathbf{Q}}$, we have*

$$h_{\mathrm{Fal}}(X) \leq 10^{25} c(B) (\deg \pi)^7, \quad \Delta(X) \leq 10^{25} c(B)(\deg \pi)^7,$$

where $c(B)$ is an explicit constant depending only on B. We can take

$$c(B) = (4 N H_B)^{45 N^3 2^{N-2} N!}.$$

Proof. This follows from the proof of Belyi's theorem. Let us be more precise. Belyi proved that, for all finite subsets $B \subset \mathbf{P}^1(\overline{\mathbf{Q}})$, there exists a rational function $R : \mathbf{P}^1 \to \mathbf{P}^1$ étale over $\mathbf{P}^1 - B$ such that $R(B) = \{0, 1, \infty\}$. Khadjavi showed that one can take R such that $\deg R \leq (4 N H_B)^{9 N^3 2^{N-2} N!}$. Now, composing π with R we obtain a Belyi map $R \circ \pi : X \to \mathbf{P}^1$. Therefore, $\deg_B(X) \leq \deg R \deg \pi$. Substituting this into the main result gives the theorem. $\square$

Remark 5.0.12. We were first led to investigate this problem by a conjecture of Edixhoven, de Jong and Schepers on the Faltings height. Their conjecture follows from the above corollary. We do not state this conjecture. A precise formulation can be found in [EdJoSc, Conjecture 5.1]. We do explain the "context" in which this conjecture plays a key role.

Remark 5.0.13. (Computing étale cohomology) Let S be a smooth projective geometrically connected surface over $\mathbf{Q}$. In view of our main result and results of Edixhoven et al., it seems **reasonable to suspect** that, following a strategy of Edixhoven, there is an algorithm that on input a prime p computes, for $i = 0, \ldots, 4$, the cohomology groups $H^i(S_{\overline{\mathbf{Q}},et}, \mathbf{F}_p)$ with their $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-action, in time polynomial in p.

Such an algorithm (polynomial or not) doesn't exist at the present time. If such an algorithm is devised following Edixhoven's strategy it will be polynomial because of our main result.

6 Application 2: Szpiro's small points conjecture

We state three very similar Diophantine conjectures.

Conjecture 6.0.14. (Belyi degree) *Let K be a number field, S a finite set of finite places of K and $g \geq 2$ an integer. Then there exists an* ***explicit*** *real number $c(K,S,g)$ such that, for any smooth projective geometrically connected curve X over K of genus g with good reduction outside S, the inequality*

$$\deg_B(X) \leq c(K,S,g)$$

holds.

Conjecture 6.0.15. (Effective Shafarevich) *Let K be a number field, S a finite set of finite places of K and $g \geq 2$ an integer. Then there exists an* ***explicit*** *real number $c(K,S,g)$ such that, for any smooth projective geometrically connected curve X over K of genus g with good reduction outside S, the inequality*

$$h_{\mathrm{Fal}}(X) \leq c(K,S,g)$$

holds.

For $x \in X(\overline{\mathbf{Q}})$, we let $h(x)$ be the "Arakelov height" of x. This is a non-negative real number with the property that, for any real number C and integer $d \geq 1$, the set

$$\{x \in X(\overline{\mathbf{Q}}) : [\mathbf{Q}(x) : \mathbf{Q}] \leq d, h(x) \leq C\}$$

is finite.

Conjecture 6.0.16. (Szpiro's small points conjecture) *Let K be a number field, S a finite set of finite places of K and $g \geq 2$ an integer. Then there exists an* ***explicit*** *real number $c(K,S,g)$ such that, for any smooth projective geometrically connected curve X over K of genus g with good reduction outside S, there exists a point $x \in X(\overline{\mathbf{Q}})$ with*

$$h(x) \leq c(K,S,g).$$

Proposition 6.0.17. *The Belyi conjecture implies the effective Shafarevich conjecture and Szpiro's small points conjecture.*

Proof. This follows from the main result and its proof. (One can also combine the main result with the arithmetic Faltings-Riemann-Roch theorem.) □

Remark 6.0.18. The above conjectures are theorems if you remove "**explicit**" from the statements.

Theorem 6.0.19. *The three above conjectures hold for cyclic covers. Let us be more precise. Let S be a finite set of places of K. Let X be a smooth projective geometrically connected curve over a number field K of genus $g \geq 2$ with good reduction outside S. Assume $X_{\overline{K}}$ is a cyclic cover of the projective line of prime degree. Then, there exists an explicit real number $c(K, S, g)$ depending only on K, S and g, and $x \in X(\overline{\mathbf{Q}})$ such that*

$$\deg_B(X) \leq c(K, S, g), \quad h_{\mathrm{Fal}}(X) \leq c(K, S, g), \quad h(x) \leq c(K, S, g).$$

Proof. It suffices to prove the inequality for $\deg_B(X)$. The idea is to use Khadjavi's bound. That is, for $\pi : X \to \mathbf{P}^1$ a rational function etale over $\mathbf{P}^1 - B$, we have

$$\deg_B(X) \leq (4NH_B)^{92^{N-2}N^3N!}.$$

Thus, it suffices to find a rational function with $\deg \pi \leq c(K, S, g)$, $N \leq c(K, S, g)$ and $H_B \leq c(K, S, g)$. It is easy to find a function $\pi : X \to \mathbf{P}^1$ such that $\deg \pi \leq g + 1$ and $N \leq 4g[K : \mathbf{Q}]$ using Riemann-Roch and Riemann-Hurwitz. It is difficult to bound H_B explicitly in terms of K, S and g in general. But, if $\pi : X \to \mathbf{P}^1$ is a cyclic cover (geometrically), then one can use Baker's theory of logarithmic forms to obtain an explicit bound on H_B in terms of K, S and g. $\square$

Remark 6.0.20. It would be interesting to prove the above conjectures in their full generality. In fact, this would imply an **effective version of the Mordell conjecture**.

7 How does one prove Theorem 4.0.9

We sketch the proof of Theorem 4.0.9. Let $X/\overline{\mathbf{Q}}$ be a curve of genus $g \geq 1$. To simplify the exposition, we will restrict ourselves to proving the following inequality:

$$e(X) \leq 3 \cdot 10^7 (g - 1) \deg_B(X)^5.$$

The first step is provided by Faltings' bound for $e(X)$ which he derived from the arithmetic Hodge index theorem.

Theorem 7.0.21. (Faltings) *For all x in $X(\overline{\mathbf{Q}})$, the inequality $e(X) \leq 4g(g - 1)h(x)$ holds.*

We will give a formula for $h(x)$ later. It is clear that Faltings' theorem shows that it suffices to prove the following result.

Theorem 7.0.22. *There exists a point x in $X(\overline{\mathbf{Q}})$ such that*

$$h(x) \leq \frac{3 \cdot 10^7}{4g} \deg_B(X)^5.$$

Let $a \in \mathbf{P}^1(\overline{\mathbf{Q}})$ and let $\pi : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ be a Belyi map. We will show that, for all $x \in \pi^{-1}(a)$, the inequality

$$h(x) \leq \frac{3 \cdot 10^7}{4g} \deg(\pi)^5$$

holds. This will finish the proof.

We assume X has a semi-stable minimal regular model $\mathcal{X}_{\mathrm{min}}$ over O_K and that x is K-rational. (This can be obtained by replacing K with a finite extension.)

We consider $d\pi$ as a rational section of $\omega_{\mathcal{X}_{min}/O_K}$. Let $K_{\mathcal{X}_{min}}$ be the (usual) Cartier divisor of $d\pi$ on $\mathcal{X}_{\mathrm{min}}$. Let $\| \cdot \|_{\mathrm{Ar}}$ be the Arakelov norm function on $\Omega^1_{X(\mathbf{C})}$ and let $(\cdot, \cdot)$ be the usual intersection pairing on $\mathcal{X}_{min}$.

Lemma 7.0.23. *For all x such that $\pi(x) \notin \mathbf{P}^1 - \{0, 1, \infty\}$, we have*

$$[K : \mathbf{Q}]h(x) = (K_{\mathcal{X}_{min}}, x) + \sum_{\sigma : K \to \mathbf{C}} -\log \|d\pi_\sigma\|_{\mathrm{Ar}}(x).$$

(Here we let x also denote the induced section of $\mathcal{X}_{min} \to \operatorname{Spec} O_K$.)

This lemma implies that, to prove our main result, it suffices to prove the following theorem.

Theorem 7.0.24. *Let $a = -1$ in $\mathbf{P}^1(\mathbf{Q})$. For all $x \in \pi^{-1}(a)$, the following inequalities hold.*

Arithmetic: *We have*

$$(K_{\mathcal{X}_{min}}, x) \leq 2 \deg_B(X)^3 [K : \mathbf{Q}];$$

Analytic: *We have*

$$\sum_{\sigma : K \to \mathbf{C}} -\log \|d\pi_\sigma\|_{\mathrm{Ar}}(x) \leq \frac{7 \cdot 10^6}{g} \deg_B(X)^5 [K : \mathbf{Q}].$$

8 The arithmetic part

We want to show that $(K_{\mathcal{X}_{min}}, x) \leq 2 \deg_B(X)^3 [K : \mathbf{Q}]$. We proceed in six steps.

1. Let $p : \mathcal{X} \to \mathbf{P}^1_{O_K}$ be the normalization of $\mathbf{P}^1_{O_K}$ in the function field of X. Then

   $$(K_{\mathcal{X}_{min}}, x) \leq (K_{\mathcal{X}}, x).$$

   The proof uses birational geometry of arithmetic surfaces.

2. The (generalized) Riemann-Hurwitz formula states that $K_{\mathcal{X}} = \pi^* K_{\mathbf{P}^1_{O_K}} + R$ as Weil divisors on $\mathcal{X}$, where R is the ramification divisor of $p : \mathcal{X} \to \mathbf{P}^1_{O_K}$ and $K_{\mathbf{P}^1_{O_K}}$ is the canonical divisor on $\mathbf{P}^1_{O_K}$ associated to the tautological section. We note that R is supported on $\pi^{-1}(D)$, where D is the branch locus of p. Moreover, if D' is an irreducible component of $\pi^{-1}(D)$, then the multiplicity of R along D' is the valuation of the different ideal of $\mathcal{O}_{D'}$ over $\mathcal{O}_{p(D')}$.

3. We deduce, using the projection formula, that

   $$(K_{\mathcal{X}}, x) \leq (a, p_* R).$$

   We recall that $a = -1$.

4. The fourth step consists of proving Lenstra's generalization of Dedekind's discriminant conjecture.

   Lemma 8.0.25. (Lenstra) *Let A be a discrete valuation ring of characteristic zero with fraction field K. Let L/K be a finite field extension of degree n, and let B be the integral closure of A in L. Then, for any prime ideal β of B, the exponent of β in the different ideal $\mathfrak{D}_{B/A}$ is less or equal to $e_\beta - 1 + e_\beta \operatorname{ord}_A(n)$, where e_β is the ramification index of β and ord_A is the valuation on A.*

   Proof. We may and do assume that B is a discrete valuation ring. Let x be a uniformizer of A. The trace of $y := \frac{1}{nx}$ is $\frac{1}{x}$. Since $1/x$ is not in A, this implies that the inverse different $\mathfrak{D}^{-1}_{B/A}$ is strictly contained in the fractional ideal yB. In particular, the different $\mathfrak{D}_{B/A}$ strictly contains the fractional ideal (nx). □

5. Now, using Lenstra's result, we deduce that

$$(a, p_* R) \leq 2(\deg \pi)^2 (D, a).$$

6. The final step consists of bounding (D, a). The idea is that $D = D^{hor} + D^{ver}$. It is not hard to show that $(D^{hor}, a) = 0$ (because $a = -1$). Thus, we reduce to bounding (D^{ver}, a). This can be achieved by applying Abhyankar's lemma ([SGA1, Exposé X, Lemme 3.6]) to eliminate tame ramification. We obtain that $(D, a) \leq \deg \pi [K : \mathbf{Q}]$. In fact, (D^{ver}, a) is simply the number of vertical components of D. We can get rid of all the vertical components lying over primes $\mathfrak{p}$ such that $\mathrm{char}(k(\mathfrak{p})) \geq \deg \pi$. Then, we bound the number of maximal ideals $\mathfrak{p} \subset O_K$ such that $\mathrm{char}(k(\mathfrak{p})) \leq \deg \pi$ by $\deg \pi [K : \mathbf{Q}]$.

To summarize: we have

$$(K_{\mathcal{X}_{min}}, x) \leq^1 (K_{\mathcal{X}}, x) \leq^{2,3} (a, p_* R) \leq^{4,5} 2d^2 \leq^6 3(\deg \pi)^3 [K : \mathbf{Q}]$$

9 The analytic part

Let X be a compact connected Riemann surface of genus g. Let $\pi : X \to \mathbf{P}^1$ be a Belyi map. To finish the proof we want to show that

$$\sup_X - \log \|d\pi\|_{\mathrm{Ar}} \leq \frac{7 \cdot 10^6}{g} (\deg \pi)^5.$$

The main ingredient is the following result of Merkl and Bruin.

Theorem 9.0.26 (Merkl-Bruin). *Let $(\{(U_j, z_j)\}_{j=1}^n, r_1, M, c_1)$ be a "Merkl atlas" for X. Then, for all $j = 1, \ldots, n$,*

$$\sup_{U_j} - \log \|dz_j\|_{\mathrm{Ar}} \leq \frac{330n}{(1 - r_1)^{3/2}} \log \frac{1}{1 - r_1} + 13.2 n c_1 + (n - 1) \log M.$$

Merkl proved this theorem without explicit constants and without the dependence on r_1. A proof of the theorem in a more explicit form was given by P. Bruin in his master's thesis. This proof is reproduced, with minor modifications, in the appendix of [J].

For the sake of completeness, we give the definition of a Merkl atlas for X. Let μ denote the Arakelov $(1,1)$-form on X.

Definition 9.0.27. A *Merkl atlas* for X is a quadruple $(\{(U_j, z_j)\}_{j=1}^n, r_1, M, c_1)$, where $\{(U_j, z_j)\}_{j=1}^n$ is a finite atlas for X, $\frac{1}{2} < r_1 < 1$, $M \geq 1$ and $c_1 > 0$ are real numbers such that the following properties are satisfied.

1. Each $z_j U_j$ is the open unit disc.
2. The open sets $U_j^{r_1} := \{x \in U_j : |z_j(x)| < r_1\}$ with $1 \leq j \leq n$ cover X.
3. For all $1 \leq j, j' \leq n$, the function $|dz_j / dz_{j'}|$ on $U_j \cap U_{j'}$ is bounded from above by M.
4. For $1 \leq j \leq n$, write $\mu_{\mathrm{Ar}} = i F_j dz_j \wedge d\overline{z_j}$ on U_j. Then $0 \leq F_j(x) \leq c_1$ for all $x \in U_j$.

10 How to construct a Merkl atlas

It's clear that we need to construct a Merkl atlas. We won't get into too much details. We will just explain how one constructs an atlas for X in a "controlled" manner. If the reader is interested in showing the atlas we construct below is a Merkl atlas with "controlled" parameters he or she may look at Section 3 of [J].

We have already mentioned the isomorphism $X(1) \cong \mathbf{P}^1(\mathbf{C})$ given by the j-invariant. There is also an isomorphism $X(2) \cong \mathbf{P}^1(\mathbf{C})$ given by the λ-invariant. We replace $\mathbf{P}^1$ with $X(2)$.

One can define charts at each cusp of $X(2)$:

$$(B_0, z_0), \quad (B_1, z_1), \quad (B_\infty, z_\infty).$$

This gives an atlas for $X(2)$. Also, these charts have the property that $z_\kappa : B_\kappa \to B(0,1)$ is an isomorphism, where $B(0,1)$ is the open unit disc in $\mathbf{C}$. Also, each B_κ contains precisely one cusp: κ.

Now, we "lift" the atlas to an atlas $\{(V_y, w_y)\}_{y\in V}$, where we write V for $\pi^{-1}(\mathbf{P}^1 - \{0,1,\infty\})$, as follows.

Let κ be a cusp of $X(2)$. The branched cover $\pi^{-1}(B_\kappa) \longrightarrow B_\kappa$ restricts to a finite degree topological cover $\pi^{-1}(\dot{B}_\kappa) \longrightarrow \dot{B}_\kappa$. In particular, the composed morphism

$$\pi^{-1}\dot{B}_\kappa \longrightarrow \dot{B}_\kappa \xrightarrow[z_\kappa|_{\dot{B}_\kappa}]{\sim} \dot{B}(0,1)$$

is a finite degree topological cover of $\dot{B}(0,1)$.

Recall that the fundamental group of $\dot{B}(0,1)$ is isomorphic to $\mathbf{Z}$. More precisely, for any connected topological cover of $V \to \dot{B}(0,1)$, there is a unique integer $e \geq 1$ such that $V \to \dot{B}(0,1)$ is isomorphic to the cover $\dot{B}(0,1) \longrightarrow \dot{B}(0,1)$ given by $x \mapsto x^e$.

For every cusp y of Y lying over κ, let $\dot{V}_y$ be the unique connected component of $\pi^{-1}\dot{B}_\kappa$ whose closure V_y in $\pi^{-1}(B_\kappa)$ contains y. Then, for any cusp y, there is a positive integer e_y and an isomorphism $w_y : \dot{V}_y \xrightarrow{\sim} \dot{B}(0,1)$ such that $w_y^{e_y} = z_\kappa \circ \pi|_{\dot{V}_y}$. The isomorphism $w_y : \dot{V}_y \longrightarrow \dot{B}(0,1)$ extends to an isomorphism $w_y : V_y \longrightarrow B(0,1)$ such that $w_y^{e_y} = z_\kappa \circ \pi|_{V_y}$. This shows that e_y is the ramification index of y over κ. Note that we have constructed an atlas $\{(V_y, w_y)\}$ for Y, where y runs over the cusps of Y.

Showing this atlas is a Merkl atlas is essentially an elementary (but tedious) computation. The only non-trivial result we use is a result of Jorgenson-Kramer-Bruin on the Arakelov (1,1)-form and the hyperbolic (1,1)-form.

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