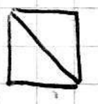
Yuri MANIN – Quantum cohomology, motives, derived categories

1. History
2. Quantum cohomology by my eyes
3. Structures that have been studied
4. dg-categories

The second oldest (math.) profession:

- solving equations
- number of solutions of eqs.
- intrinsic structure

$x^2 = 2$  $\Rightarrow$ irrational numbers

$x^2 = -1$ $\Rightarrow$ imaginary numbers

π $\Rightarrow$ transcendental

$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ solutions can coincide

$x^n = 0$ $\leftarrow$ multiplicity

$$\begin{cases} f_m(x,y) = 0 \\ g_n(x,y) = 0 \end{cases}$$

$mn \approx$ generic case

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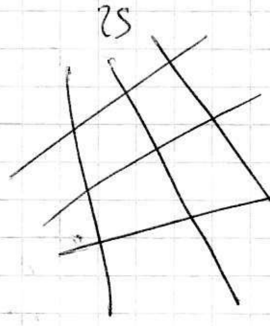

easiest way to study the generic case = produce a very degen. case

correct multiplicity:

- proj. plane
- no common components

clever degeneration

Serre:

$$\mathrm{mult}_p = \sum (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_p}(\mathcal{O}_{p,f=0}, \mathcal{O}_{p,g=0})$$

if the answer is not correct, the way to get the answer should be corrected

A bit of number theory: Jacobi (1829)

$$r_4(\underset{\substack{\uparrow \\ \text{variable}}}{n}) := \mathrm{Card}\left\{ (j_i) \in \mathbb{Z}^4 \;\middle|\; \sum_{i=1}^{4} j_i^2 = n \right\}$$

Theorem

$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d$$

Interest for many reasons: 2 diff. programs to compute the same function

(1) elementary proof using combinatorics

(2) computation

Proof:

$$\sum_{n=0}^{\infty} (-1)^n r_4(n) q^n$$

$$= \Big(\sum_{-\infty < j < \infty} (-1)^j q^{j^2} \Big)^4 \overset{(!)}{=} 1 - 8 \sum_{d=1}^{\infty} (-1)^{d-1} \frac{d q^d}{1+q^d}$$

$\square$

$q = e^{2\pi i z}$, $\mathrm{Im}(z) > 0$

algebraic function on moduli space of elliptic curves

Candelas - de la Ossa - Green, Parkes (1991)

$n(d)$ = "virtual" number of rational (parametrized) curves of degree d on a "generic" 3-dimensional quintic $V_5 \subset \mathbb{P}^4$.

$$f : \mathbb{P}^1 \to \mathbb{P}^4 \text{ of deg } d \; / \; \text{autom } \mathbb{P}^1$$

how many of them lie in V_5?

$$F(y) := \frac{5}{6} y^3 + \sum_{d=1}^{\infty} n(d) e^{dy}$$

Mirror formula:

$$F'''(y)\Big|_{y = \frac{\Psi_1}{\Psi_0}} = \frac{5}{2} \frac{\Psi_1 \Psi_2 - \Psi_0 \Psi_3}{\Psi_0^2}$$

$$\Psi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$$

$$\Psi_1(z) = \log z \cdot \Psi_0(z) + \cdots$$

$$\Psi_2(z) = \frac{1}{2} (\log z)^2 \cdot \Psi_0(z) + \cdots$$

$$\partial = z \frac{\partial}{\partial z}$$

$$\left[\partial^4 - 5z(5\partial+1)(5\partial+2)(5\partial+3)(5\partial+4)\right] \Psi(z) = 0$$

Protagonists

- All smooth, projective varieties $V, X \dots$ over $\mathbb{C}$ [DM-stacks]

  each point may have a finite autom. group

- A very interesting subclass of them: $\overline{M}_{g,n}$

$$\begin{cases} g = 0, 1, 2, \dots \\ \{1, 2, \dots n\} = \text{labels on a set of } n \text{ pairwise} \end{cases}$$

different points on a <u>curve of genus</u> g

can be reducible & singular

+ stability condition

 but "mild"

(Bridgeland: "moduli spaces" of stability)

$\# \mathrm{Aut}(C, (x_1, \dots x_n)) < \infty$

e.g. in genus 0, $n \geq 3$

$\overline{M}_{g,n}$:= the moduli (stack) space of stable curves $(C, (x_1, \dots x_n))$.

- $\{\overline{M}_{g,n}\}$ + a set of canonical morphisms = a structure of <u>operad</u>

- The basic result of quantum cohomology:

Operad of $\{\overline{M}_{g,n}\}$ "<u>acts upon any</u> V" so that it forms an algebra

In topology: Steenrod alg. of universal operations acting on all top. spaces

Explorations: action is on the level of motives

$\Longleftrightarrow$ $\overline{M}_{g,n} \times V^n$ there is a correspondence

$$I_{g,n}(V) \in A^*(\overline{M}_{g,n} \times V^n)$$

- In fact, this correspondence depends on one more parameter $\beta \in A_1(V)$

$$I_{g,n}(V, \beta) \quad \text{functoriality wrt } V \text{ and } \beta$$

$\rightsquigarrow$ acts upon $\coprod_{\beta \in A_1(V)} V$

operadic structure of $\overline{M}_{g,n}$ translates into relations in this algebra...

final statement is alg. geom but you cannot move things to a generic situation inside alg. geom. $\rightsquigarrow$ need to go to symplectic geometry.

Open question: is the number $n(d)$ attained?

V, $\beta \in A_1^{num}(V)$ numerical class of effective curves

g, n

Contract

$f_*([C]) = \beta$

space of maps

$$\begin{tikzcd}[column sep=small] & \overline{M}_{g,n}(V,\beta) \arrow[dl, "st"'] \arrow[dr, "ev"] & \\ \overline{M}_{g,n} & & V^n \end{tikzcd}$$

$$(C, (x_1, \ldots x_n)) \xrightarrow[f]{\text{stable}} V$$

autom. acting identically on $(x_1, \ldots x_n)$ & V should be finite

$$\begin{tikzcd}[column sep=small] & (C,(x_1,\ldots x_n)) \xrightarrow{f} V \arrow[dl, "\text{forget } f \text{ and stabilise}"'] \arrow[dr] & \\ (C, (x_1, \ldots x_n))^{st} & & (f(x_1), \ldots f(x_n)) \in V^n \end{tikzcd}$$

Tentatively: $(st, ev)_* \underbrace{[\overline{M}_{g,n}(V,\beta)]}_{\substack{\cap \\ A_*(\overline{M}_{g,n}(V,\beta))}}^{\text{virt}} \in A_*(\overline{M}_{g,n} \times V^n)$

no axiomatic definition of the virtual fund. class even if it is extremely universal!

$$\left\{ \begin{array}{l} I_{g,n}(V,\beta) \in A_*(\overline{M}_{g,n} \times V^n) \\ \quad \uparrow \\ \mathcal{I}_{g,n}(V,\beta) \in A_*(\overline{M}_{g,n}(V,\beta)) \end{array} \right.$$

several contractions of these classes

$\downarrow$

F-manifolds $\leftarrow$ $\boxed{\text{properties}}$

mum identities $\leftarrow$ + (P) $\leftarrow$

• Metaproblem: find a list of properties uniquely defining these correspondences

• Motives: a linear tensor category "linearizing" $\underline{Var}$, $\otimes$, Ker + coker of projectors

Tannakian properties

Program of Deligne

Vec	Tann
$Sym^n V$	

$\Downarrow$

Algebraic geometry over Mot_k. $\overset{?}{\Longrightarrow}$ Analytic geometry ??

arithmetic identities with motivic interpretations of Jacobi, de la Vra...

2 power series are the same after some transformations :

grading

Classical motives

Var_k (e.g. $k = \mathbb{C}$)

• Adequate intersection theory : (i) $X \mapsto A^*(X)$

modulo a certain equivalence relation

$\mathbb{Z}$, $\mathbb{Q}$, $\overline{\mathbb{Q}}$, $\mathbb{C}$ ...

coefficient ring

numerical

~~homological~~

algebraic

rational

(ii) $f : X \to Y$

$\Rightarrow f^* : A^*(Y) \to A^*(X)$

$f_* : \ldots$

(iii) $A^*(X) \otimes A^*(Y) \to A^*(X \times Y)$

but looking only at alg. cycles can we capture the transcendental cohomology?

(iv) $\Delta : X \to X \times X$

$$\Delta^* : A^*(X) \otimes A^*(X) \to A^*(X \times X) \xrightarrow{\Delta^*} A(X)$$

comm. ass. ring

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y$$

$\Longrightarrow \quad \mathrm{Var}_k \Rightarrow A^*$ – correspondences

$$\mathrm{Corr}^r(X, Y) = A^{d+r}(X^d \times Y) \qquad \dim X = d$$

$$\mathrm{Corr}^r(X, Y) \times \mathrm{Corr}^s(Y, Z) \to \mathrm{Corr}^{r+s}(X, Z)$$

$$f \otimes g \longmapsto g \circ f := p_{XZ*}(p_{XY}^*(f) \cdot p_{YZ}^*(g))$$

<u>A^*-correspondences</u>:

objects: $[V]$

morphisms: $\mathrm{Corr}^*(X, Y)$

$$\underbrace{X \times Y}_{f} \times Z \qquad X \times \overbrace{Y \times Z}^{g}$$

$$h : \mathrm{Var}_k^{opp} \to A\text{-}\mathrm{Corr}^\circ \to \mathrm{Mot}$$

[finally add images/ker of projectors]

<u>Mot</u> objects: $([X], q, n)$

$$q^2 = q$$

$$\mathrm{Hom}((Y, q, n), (X, p, m)) = q \circ \mathrm{Corr}^{n-m}(X \times Y) \circ p$$

$$\cap$$

$$\mathrm{Corr}^*(X, Y)$$

unique Tate motive $\mathbb{L}$:

$$[\mathbb{P}^n] = \underset{\text{point}}{\mathbb{1}} \oplus \underset{\mathbb{A}^1}{\mathbb{L}} \oplus \dots \oplus \underset{\mathbb{A}^n}{\mathbb{L}^{\otimes n}}$$

$(\mathrm{Mot}^*_k, \otimes, \text{linear})$

$$\mathrm{Var}_k \xrightarrow{h} \mathrm{Mot}_k$$

$$X \mapsto (X, id_X, 0)$$

$\overline{M}_{g,n}$ have themselves quantum cohomology & they act upon themselves: case $\overline{M}_{0,n}$

Quantum cohomology of genus 0 and (formal) Frobenius manifolds

$V/\mathbb{C} \Rightarrow H^*(V, K)$ "simplest representative of $\mathbb{Z}/2$-grading the motive of V":

$\left\{ \begin{array}{l} \text{interesting } \underline{\text{characteristic functions}} \text{ that can be} \\ \text{constructed from } I^\beta_{0,n}(\overline{M}_{0,n} \times V^n),\ \beta \in A_1^{num,+}(V) \end{array} \right\}$

encode $\Longrightarrow$ a formal geometric structure

formal completion of

$$H^*(V, K)[\![q^\beta]\!] \quad \beta \in A_1^{num,+}$$

$$\sum_{\text{cycles}} \sum_{\beta} \left\{ \begin{array}{l} \text{int. indices involving} \\ \text{cycles on } V \text{ and } \beta \\ \text{and } I^\beta_{0,n} \end{array} \right\} q^\beta$$

Structure of Frobenius manifolds

$(M, (T_M, \circ, e))$ ← identity

T_M: tangent sheaf

$\circ$: comm. assoc. mult. bilinear over $\mathcal{O}_M$

a "metric" $g \in S^2(T_M^*)$

+ a system of axioms expressing the compatibility of structures

(in the category of supermanifolds)
fermion coordinates

$(M, (T, \circ, e))$ + compatibility (F-manifold)

In symplectic geometry:

$(M, (\mathcal{O}_M, \cdot), (T_M, [\,,\,]))$ → F-manifolds $(M, (T_M, [\cdot\,], \circ))$

$M, (\mathcal{O}_M, \cdot, \{\,,\,\})$

classical compatibility

F-compatibility?

$\operatorname{Spec} S^*(T_M) \simeq$ cotangent manifold to M — canonically symplectic

$\downarrow$

M

$$S^*(T_M) \xrightarrow{\circ} T_M$$
$$\cup$$
$$\mathrm{Ker}$$

genetic explanation

$[\mathrm{Ker}, \mathrm{Ker}] \subset \mathrm{Ker}$ ⟵ compatibility condition

Are there interesting compact F-manifolds?

Take $V = \overline{M_{0,n}}$

don't know the structure of the core of β

The correct definition of $I^{\beta}_{0,n}$ requires heavy cohomological algebra.

Modern theory of motives starts with the suggestion that

MOTIVE of V is represented by $\mathrm{Der}^b(V)$

$\sim$ DG - version

M. Kontsevich, Tabuada (21st century version)

objects: dg-categories with some compatibility

derived functor

complex ⟺ formal power series