An overview of Derived Algebraic Geometry

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Plan of the talk

1. Motivations
2. An overview of the theory
3. Developments and applications
   - Reduced obstruction theory for stable maps to a K3
   - Derived symplectic structures and quantized moduli spaces

Who?

'pre-history' – V. Drinfel'd, P. Deligne, M. Kontsevich, C. Simpson

'history' – B. Toën - G.V., J. Lurie

current – (the above ones and) M. Vaquié, T. Schürg, C. Barwick, D. Spivak, T. Pantev, D. Calaque, L. Katzarkov, D. Gaitsgory, D. Joyce, C. Brav, V. Bussi, D. Borisov, J. Noel, J. Francis, A. Preygel, N. Rozenblyum, O. Ben Bassat, J. Wallbridge, A. Blanc, M. Robalo, E. Getzler, K. Behrend, P. Pandit, B. Hennion, S. Bach, V. Melani, M. Porta, M. Cantadore, and many more (sorry for possible omissions) ...

Pretty much a collective activity !

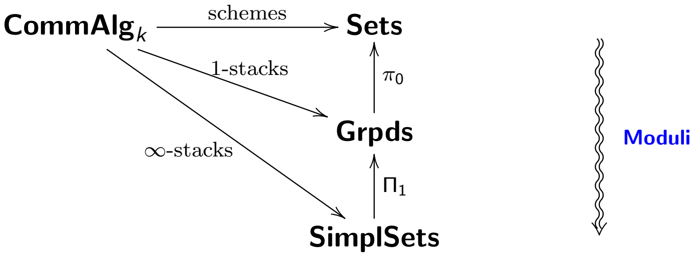
Why derived geometry (historically)?

Motivations form Algebraic Geometry and Topology :

- Hidden smoothness philosophy (Kontsevich): singular moduli spaces are truncations of 'true' moduli spaces which are smooth (in some sense) $\rightsquigarrow$ good intersection theory.
- Understand more geometrically and functorially obstruction theory and virtual fundamental class (Li-Tian, Behrend-Fantechi), and more generally deformation theory for schemes, stacks etc. (e.g. give a geometric interpretation of the full cotangent complex, a question posed by Grothendieck in 1968 !).
- Conjecture on elliptic cohomology (V, $\sim$ 2003; then proved and vastly generalized by J. Lurie): Topological Modular Forms (TMF) are global sections of a natural sheaf on a version of $\mathcal{M}_{\mathrm{ell}} \equiv \overline{\mathcal{M}}_{1,1}$ defined as a derived moduli space modeled over commutative (a.k.a E_∞) ring spectra.
- Realize C^∞-intersection theory without transversality $\rightsquigarrow$ C^∞-derived cobordism (achieved by D. Spivak (2009)).

A picture of (underived) Algebraic Geometry

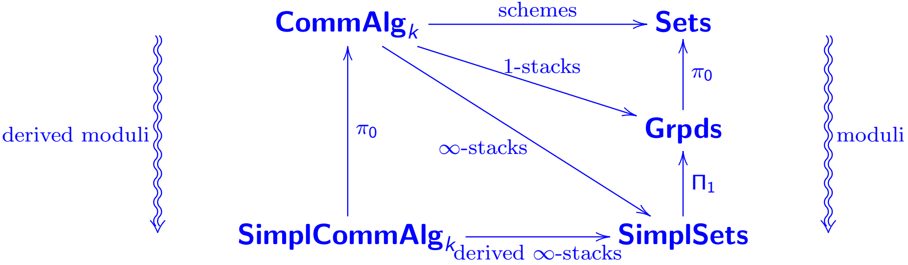
Schemes, algebraic spaces $\rightsquigarrow$ 1-stacks $\rightsquigarrow$ ∞-stacks
The functor of points point of view is :



Moduli extension of target categories: allows taking quotients properly, and classifying geometric objects up to a more general notion of equivalences (not only isos) ($\equiv$ adjoining homotopy colimits)
$\Rightarrow$ promotes the target categories to a full homotopy theory (that of **SimplSets** or, equivalently, of topological spaces).

A picture of Derived Algebraic Geometry (DAG)

If we 'extend' also the target category $\Rightarrow$



Derived Algebraic Geometry: both source and target are nontrivial homotopy theories. (Roughly: up-to-htpy sheaves on up-to-htpy coverings.)

Over a base commutative $\mathbb{Q}$-algebra k we may (and will !) replace the category of derived affine objects **SimplCommAlg**$_k$ with **cdga**$_k$, i.e. commutative differential nonpositively graded k-algebras (cdga's)

$$\ldots \xrightarrow{d} A^{-2} \xrightarrow{d} A^{-1} \xrightarrow{d} A^{0} \longrightarrow 0$$

Why should DAG be like that ?

We will motivate the appearance of cdga's in the previous picture, in two ways:

- via hidden smoothness
- via 'fixing' the naturality of classical deformation theory

Actually, these motivations are strictly related, but for the sake of the presentation ...

I - Motivating DAG through hidden smoothness

X - smooth projective variety $/\mathbb{C}$
Vect$_n(X)$: moduli stack classifying rank n *vector bundles* on X
$x_E : \mathrm{Spec}\, \mathbb{C} \to \mathbf{Vect}_n(X) \Leftrightarrow E \to X \Rightarrow$ stacky tgt space is

$$T_{x_E}\mathbf{Vect}_n(X) \simeq \mathbb{R}\Gamma^{\leq 1}(X_{\mathrm{Zar}}, \mathrm{End}(E))[1]$$

• If dim $X = 1$ there is no truncation $\Rightarrow$ dim T_E is locally constant $\Rightarrow$ **Vect**$_n(X)$ is smooth.
• if dim $X \geq 2$, truncation is effective $\rightsquigarrow$ dim T_E is not locally constant $\Rightarrow$ **Vect**$_n(X)$ is not smooth (in general).

Upshot : smoothness would be assured for any X, if **Vect**$_n(X)$ was a 'space' with tangent complex the full $\mathbb{R}\Gamma^{\leq 1}(X_{\mathrm{Zar}}, \mathrm{End}(E))[1]$ (i.e. no truncation).
But : (for arbitrary X) $\mathbb{R}\Gamma(X_{\mathrm{Zar}}, \mathrm{End}(E))[1]$ is a perfect complex in arbitrary positive degrees $\Rightarrow$ it cannot be the tangent space of any 1-stack (nor of any n-stack for any $n \geq 1$).

I - Motivating DAG through hidden smoothness

So we need a new kind of spaces to accommodate tangent spaces $\mathbb{T}$ in degrees $[0, \infty)$.

To guess heuristically the local structure of this spaces

- require smoothness (i.e. uncover hidden smoothness)
- then, locally at any point, should look like $\mathrm{Spec}(\mathrm{Sym}(\mathbb{T}^\vee))$

$\Rightarrow$ local models for these spaces are **cdga**'s i.e. commutative differential graded $\mathbb{C}$-algebras in degrees ≤ 0 (equivalently, simplicial commutative $\mathbb{C}$-algebras) and $\mathbb{T}$ is only defined up to quasi-isomorphisms (isos in cohomology).

Upshot : local/affine objects of derived algebraic geometry should be cdga's defined up to quasi-isomorphism.

II - Motivating DAG through deformation theory

Derived deformation theory ($:=$ deformation theory in DAG) fills the 'gaps' in classical deformation theory ($k = \mathbb{C}$ here).

Moduli problem:

$F : \mathsf{commalg}_{\mathbb{C}} \longrightarrow \mathrm{Grpds} \,:\, R \mapsto \{ Y \to \mathrm{Spec}\, R \,,\ \mathrm{proper}\ \&\ \mathrm{smooth} \}$

Fixing $\xi = (f : X \to \mathrm{Spec}\,\mathbb{C}) \in F(\mathbb{C}) \rightsquigarrow$

Formal moduli problem : $\widehat{F}_{\xi}(A) := \mathit{hofiber}(F(A) \to F(\mathbb{C}); \xi)$, i.e.

$\hat{F}_{\xi} : \mathsf{Artin}_{\mathbb{C}} \longrightarrow \mathrm{Grpds}$

$A \mapsto \{ Y \to \mathrm{Spec}\, A \,,\ \mathrm{proper}\ \&\ \mathrm{smooth}\ +\ \mathrm{iso}\, X \simeq Y \times_A \mathbb{C} \}$

Classical deformation theory:

1. $\widehat{F}_{\xi}(\mathbb{C}[t]/t^{n+1})$ groupoid of infinitesimal n-th order deformations of ξ
2. if $\xi_1 \in \widehat{F}_{\xi}(\mathbb{C}[\varepsilon] = \mathbb{C}[t]/t^2)$, then $\mathrm{Aut}_{\widehat{F}_{\xi}(\mathbb{C}[\varepsilon])}(\xi_1) \simeq H^0(X, T_X)$
3. $\pi_0(\widehat{F}_{\xi}(\mathbb{C}[\varepsilon])) \simeq H^1(X, T_X)$
4. If $\xi_1 \in \widehat{F}_{\xi}(\mathbb{C}[\varepsilon]$, $\exists\, \mathit{obs}(\xi_1) \in H^2(X, T_X)$ which vanishes iff ξ_1 extends to a 2^{nd} order deformation $\xi_2 \in \widehat{F}_{\xi}(\mathbb{C}[t]/t^3)$.

II - Motivating DAG through deformation theory

Critique of obstructions:

1. what is the deformation theoretic interpretation of the whole $H^2(X, T_X)$?
2. how to determine the subspace of obstructions inside $H^2(X, T_X)$?

These questions

- are important classically: often $H^2(X, T_X) \neq 0$ but $\{\text{obstructions}\} = 0$ (e.g. X smooth surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree ≥ 6);
- have no answers inside classical deformation theory

Let us see how how derived algebraic geometry answers to both.

II - Motivating DAG through deformation theory

Extend the functor $F : \mathsf{commalg}_{\mathbb{C}} \longrightarrow \mathrm{Grpds}$ to a

Derived Moduli problem (derived stack):
$\mathbb{R}F : \mathsf{cdga}_{\mathbb{C}} \longrightarrow \mathrm{Grpds} \hookrightarrow \mathrm{SSets}$
$A^\bullet \mapsto \{ Y \to \mathbb{R}\mathrm{Spec}\, A^\bullet \text{, proper \& smooth} \}$

then: $\mathbb{R}F$ commutes with h-pullbacks, and $\mathbb{R}F(R) \simeq F(R)$ for $R \in \mathsf{commalg}_{\mathbb{C}} \hookrightarrow \mathsf{cdga}_{\mathbb{C}}$.

Derived formal moduli problem (formal derived stack):
$\widehat{\mathbb{R}F}_{\xi} := \mathbb{R}F \times_{\mathrm{Spec}\,\mathbb{C}} \xi : \mathsf{dgArtin}_{\mathbb{C}} \longrightarrow \mathsf{sSets}$
$\widehat{\mathbb{R}F}_{\xi}(A^\bullet) := \mathit{hofiber}(\mathbb{R}F(A^\bullet) \to \mathbb{R}F(C); \xi)$

where $\mathsf{dgArtin}_{\mathbb{C}} := \{A^\bullet \in \mathsf{cdga}_{\mathbb{C}} \mid H^0(A^\bullet) \in \mathsf{Artin}_{\mathbb{C}}\}$

II - Motivating DAG through deformation theory

Anwer to Question 1: what is the deformation theoretic interpretation of the whole $H^2(X, T_X)$?

Proposition

There is a canonical isomorphism $\pi_0(\widehat{\mathbb{R}F}_\xi(\mathbb{C} \oplus \mathbb{C}[1])) \simeq H^2(X, T_X)$

I.e. $H^2(X, T_X)$ classifies derived deformations over $\mathbb{R}\mathrm{Spec}(\mathbb{C} \oplus \mathbb{C}[1])$!
(derived deformations := deformations over a derived base).
This also explains why classical deformation could not answer this question.

Derived def-theory explains classical def-theory

Anwer to Question 2: how to determine the subspace of obstructions inside $H^2(X, T_X)$?

Lemma

The following (obvious) diagram is h-cartesian

$$\begin{array}{ccc} \mathbb{C}[t]/t^3 & \longrightarrow & \mathbb{C}[\varepsilon] = \mathbb{C}[t]/t^2 \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} \oplus \mathbb{C}[1] \end{array}$$

II - Motivating DAG through deformation theory

So

$$\begin{array}{ccc} F(\mathbb{C}[t]/t^3) & \longrightarrow & F(\mathbb{C}[\varepsilon]) \\ \downarrow & & \downarrow \\ F(\mathbb{C}) & \longrightarrow & \mathbb{R}F(\mathbb{C}\oplus\mathbb{C}[1]) \end{array}$$

is h-cartesian; this diagram maps to $F(\mathbb{C})$, and the h-fibers at ξ yields

$$\begin{array}{ccc} \widehat{F}_\xi(\mathbb{C}[t]/t^3) & \longrightarrow & \widehat{F}_\xi(\mathbb{C}[\varepsilon]) \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \widehat{\mathbb{R}F}_\xi(\mathbb{C}\oplus\mathbb{C}[1]) \end{array}$$

h-cartesian of pointed simplicial sets.

II - Motivating DAG through deformation theory

Hence, get an exact sequence of vector spaces

$$\pi_0(\widehat{F}_\xi(\mathbb{C}[t]/t^3)) \longrightarrow \pi_0(\widehat{F}_\xi(\mathbb{C}[\varepsilon])) \xrightarrow{obs} \pi_0(\widehat{\mathbb{R}F}_\xi(\mathbb{C} \oplus \mathbb{C}[1])) \simeq H^2(X, T_X)$$

Therefore : a 1st order deformation $\xi_1 \in \pi_0(\widehat{F}_\xi(\mathbb{C}[\varepsilon]))$ of ξ, extends to a 2nd order deformation $\xi_2 \in \pi_0(\widehat{F}_\xi(\mathbb{C}[t]/t^3))$ iff the image of ξ_1 vanishes in $H^2(X, T_X)$.

So, Question 2 : how to determine the subspace of obstructions inside $H^2(X, T_X)$?

Answer: The subspace of obstructions is the image of the map *obs* above.

So, in particular, classical obstructions **are** derived deformations.

Exercise: extend this argument to all higher orders infinitesimal deformations.

Derived affine schemes and homotopy theory

The upshot of our discussion so far is that :

derived affine schemes are given by cdga's and have to be considered up to quasi-isomorphisms: i.e. we want to glue them along quasi-isomorphisms not just isomorphisms. (Recall that a scheme is built out of affine schemes glued along isomorphisms.)

So we need a theory enabling us to treat quasi-isomorphisms on the same footing as isomorphisms, i.e. to make them essentially invertible. (Why 'essentially'? Formally inverting q-isos is too rough for gluing purposes - e.g. derived categories or objects in derived categories of a cover do not glue!)

Thanks to Quillen, we know a way to do it properly:
cdga's together with q-isos constitute a homotopy theory (technically speaking, a Quillen model category structure).

Derived affine schemes and homotopy theory

What is a 'homotopy theory' ? Roughly, a category M together with a distinguished class of maps w in M, called weak equivalences , such that we can define not only Hom-set (between objects in M) up to maps in w (i.e. $Hom_{w^{-1}M}(-,-)$) but a whole mapping space (top. space or simpl. set) of maps up to maps in w (i.e. $Map_{(M,w)}(-,-)$)

Examples of homotopy theories

- $(M = \mathbf{Top}, w = \text{weak homotopy eq.ces})$ and $(M = \mathbf{SimplSets}, w = \text{weak homotopy eq.ces})$
- k: comm. ring, $(\mathbf{Ch}_k, w = \text{q-isos})$ (here π_i's of mapping spaces are the Ext-groups)
- $(\mathbf{cdga}_k, w = \text{q-isos})$ $(\text{char}\, k = 0)$ $(\mathbf{SimplCommAlg}_k, w = \text{weak htpy eq.ces})$ (any k).

$w^{-1}M := \mathrm{Ho}(M)$: homotopy category of the htpy theory (M, w).
But the htpy theory (M, w) strictly enhance $\mathrm{Ho}(M)$!

DAG in two steps

Recall - A scheme, algebraic space, stack etc. is a functor

$$\mathbf{CommAlg}_k \longrightarrow \mathbf{sSets}$$

as above, that moreover

- satisfies a sheaf condition (descent) with respect to some chosen topology defined on commutative algebras
- admits a (Zariski, étale, flat, smooth) atlas of affine schemes

Example - A functor $X : \mathbf{CommAlg}_k \longrightarrow \mathbf{Sets}$ is a scheme iff

- is an étale sheaf: for any comm. k-algebra A, for any étale covering family $\{A \to A_i\}_i$ of A, the canonical map

  $$X(A) \longrightarrow \mathrm{lim}_j X(A_j)$$

  is a bijection;
- it admits a Zariski atlas $\coprod_i U_i \to X$ ($U_i = \operatorname{Spec} R_i$, $R_i \in \mathbf{CommAlg}$).

DAG in two steps

To translate this into DAG, we thus need **two steps**

- we first need a notion of derived topology and derived sheaf theory
- then we need to make sense of (Zariski, étale, flat, smooth) derived atlases.

Just as schemes, algebraic spaces and stacks are (simplicial) sheaves admitting some kind of atlases,
the first step will give us up-to-homotopy (simplicial) sheaves, among which the second step will single out the derived spaces studied by derived algebraic geometry.

DAG - 1^{st} step: derived sheaf theory

First step (Toën-V., 2004) – develop a sheaf theory over homotopy theories (= Quillen model categories) having a up-to-homotopy topology $\rightsquigarrow$ homotopy/higher topoi (model topoi in HAG I; then reconsidered and generalized further by J. Lurie)

Derived topology on a homotopy theory/model category $(M, w) \Rightarrow$ Grothendieck topology on $\mathrm{Ho}(M) = w^{-1}M$.

Examples of homotopy theories we consider

- Simplicial commutative k-algebras (k any commutative ring)
- differential graded commutative k-algebras ($\mathrm{char}\, k = 0$)

DAG - 1^{st} step derived sheaf theory

Étale derived topology on $\mathbf{dAff}_k := \mathbf{SimplCommAlg}_k^{op}$:
$\{A \to B_i\}$ is an étale covering family for derived étale topology if

- $\{\pi_0 A \to \pi_0 B_i\}$ is an étale covering family (in the usual sense)
- for any i and any $n \geq 0$, $\pi_n A \otimes_{\pi_0 A} \pi_0 B_i \to \pi_n B_i$ is an isomorphism

The intuition is:

- everything is as usual on the classical part/truncation $\pi_0(-)$,
- on the higher π_n's everything is just a pullback along $\pi_0 A \to \pi_0 B$

Rmk. This is not an ad hoc definition: it is an elementary characterization of a more conceptual definition (via derived infinitesimal lifting property).

DAG - 1^{st} step: derived sheaf theory

The choice of a derived topology (e.g. étale) on

$$\mathbf{dAff}_k := \mathbf{SimplCommAlg}_k^{op}$$

Homotopy theory of derived stacks

- induces a homotopy theory (Quillen model structure) on the category of simplicial presheaves

  $$\mathbf{dSPr}_k := \mathrm{Functors}\ \ \mathbf{SimplCommAlg}_k = \mathbf{dAff}_k^{op} \to \mathbf{SimplSets}$$

  with W:= {weak equivalences between derived stacks} given by $f : F \to G$ inducing $\pi_i(F, x) \simeq \pi_i(G, f(x))$ for any $i \geq 0$ and any x, as sheaves on the usual site $\mathrm{Ho}(\mathbf{dAff}_k)$.
- The homotopy theory of derived stacks is $(\mathbf{dSPr}_k, W)$, and $\mathbf{dSt}_k := \mathrm{Ho}(\mathbf{dSPr}_k) = W^{-1}\mathbf{dSPr}_k$.

DAG - 1^{st} step: derived sheaf theory

Therefore, a derived stack, i.e. an object in $\mathbf{dSt}_k$, is a functor $F : \mathbf{SimplCommAlg}_k \to \mathbf{SimplSets}$ such that

- F sends weak equivalences in $\mathbf{SimplCommAlg}_k$ to weak equivalences in $\mathbf{SimplSets}$
- F has descent with respect to étale homotopy-hypercoverings , i.e.

$$F(A) \to \mathrm{holim} F(B_\bullet)$$

  is an iso in $\mathrm{Ho}(\mathbf{SimplSets})$, for any A and any étale h-hypercovering $B_\bullet$ de A

Rmk. Don't worry about hypercoverings, just think of Čech nerves associated to covers in the given topology.

DAG - 1^{st} step: derived sheaf theory

- Derived Yoneda:

$$\mathbb{R}\mathrm{Spec} : \mathbf{SimplCommAlg}_k \to \mathbf{dSt}_k, \; A \mapsto \mathrm{Map}_{\mathbf{SimplCommAlg}_k}(A, -)$$

  is fully faithful (up to homotopy).
- $\mathbf{dSt}_k$ has mapping spaces $Map_{\mathrm{dSt}_k}(F, G) \in \mathbf{SimplSets}$
- $\mathbf{dSt}_k$ has internal Hom's, denoted by $MAP_{\mathrm{dSt}_k}(-, -)$:

$$Map_{\mathrm{dSt}_k}(F, \mathrm{MAP}_{\mathbf{dSt}_k}(G, H)) \simeq Map_{\mathrm{dSt}_k}(F \times G, H)$$

  and also homotopy limits and colimits, e.g. the homotopy fiber product of derived affines is

$$\mathbb{R}\mathrm{Spec}B \times^h_{\mathbb{R}\mathrm{Spec}A} \mathbb{R}\mathrm{Spec}C \simeq \mathbb{R}\mathrm{Spec}(B \otimes^{\mathbb{L}}_A C).$$

DAG - 2^{nd} step: derived geometric stacks

Smooth maps between simplicial commutative algebras:
$A \to B$ smooth if $\pi_0 A \to \pi_0 B$ is smooth and $\pi_n A \otimes_{\pi_0 A} \pi_0 B \simeq \pi_n B$ for any $n \geq 0$

Geometric types of derived stacks

F a derived stack

- A derived atlas for F is a map $\coprod_i \mathbb{R}\mathrm{Spec}\, A_i \to F$ surjective on π_0 (and satisfying some representability conditions)
- if this map is smooth (resp. étale, Zariski) we have a derived Artin stack (resp. Deligne-Mumford stack, scheme)
- The truncation preserves the type of the stack.

Using atlases (and representability) $\rightsquigarrow$ extend notion of smooth, étale, flat, etc. to maps between geometric derived stacks (as one does in the theory of underived 1-stacks)

DAG - Main properties

- There is a truncation/inclusion adjunction:

$$\mathbf{dSt}_k \underset{i}{\overset{t_0}{\rightleftarrows}} \mathbf{St}_k$$

  - i is fully faithful (hence usually omitted in notations)
  - $t_0(\mathbb{R}\mathrm{Spec} A) = \mathrm{Spec}\, \pi_0 A$
  - the adjunction map $i(t_0 X) \hookrightarrow X$ is a closed immersion

Geometric intuition: X infinitesimal or formal thickening of its truncation $t_0(X)$, (as if $t_0(X)$ was the 'reduced' subscheme of X).
In particular, the small étale sites of X and $t_0(X)$ are equivalent.

DAG - Main properties

- The inclusion $i : \mathbf{St}_k \hookrightarrow \mathbf{dSt}_k$ preserves homotopy colimits but not homotopy limits nor internal MAP $\Rightarrow$ derived tangent spaces and derived fiber products of i(schemes) are not the scheme-theoretic tangent spaces and fiber products .
- Consequence: if we define the derived tangent stack to a derived stack X naturally as $\mathbb{T}X := \mathrm{MAP}_{\mathsf{dSt}_k}(Spec\, k[\epsilon], X)$

Geometric interpretation of cotangent complex of a scheme

Y (underived) scheme $\Rightarrow \mathbb{T}i(Y) \simeq \mathbb{R}\mathrm{Spec}_Y(\mathrm{Sym}_{\mathcal{O}_Y}(\mathbb{L}_Y))$, where $\mathbb{L}_Y$ is Grothendieck-Illusie cotangent complex of Y

$\Rightarrow$ the full cotangent complex is uniquely geometrically characterized (this answers Grothendieck's question in *Catégories cofibrées additives et complexe cotangent relatif*, 1968).

DAG - Main properties

- Geometric derived stacks have a cotangent complex (representing derived derivations), and this enjoys a universal property $\Rightarrow$ it is computable. $\Rightarrow$ Deformation theory is functorial and 'easy' in DAG. We'll see an instance of this in a few slides.
- For a h-cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ S' & \xrightarrow{f} & S \end{array}$$

  the base-change formula

$$g^* \circ f_* \simeq f'_* \circ g'^*$$

  for quasi-coherent coefficients is true even if g is not flat (e.g. for a diagram of q-compact derived schemes).

DAG - Main properties

- If $\mathrm{MAP}_{\mathsf{dSt}_k}(X, Y)$ is a derived Artin stack (e.g. X flat scheme and Y Artin stack loc. finite type), and $x_f : \operatorname{Spec} k \to \mathrm{MAP}_{\mathsf{dSt}_k}(X, Y)$ is a global point, corresponding to a map $f : X \to Y$, then

$$\mathbb{T}_{x_f}\mathrm{MAP}_{\mathsf{dSt}_k}(X, Y) \simeq \mathbb{R}\Gamma(X, f^*\mathbb{T}_Y)$$

DAG - An example: derived stack of vector bundles

For $A \in \mathbf{cdga}_{\mathbb{C}}$, let $\mathbf{Mod}^{\mathrm{der}}(X, A)$ category with objects presheaves of $\mathcal{O}_X \otimes A$-dg-modules on X and morphisms inducing quasi-isomorphisms on stalks (these maps are called equivalences). Consider the functor

$$\mathbb{R}\mathbf{Vect}_n : \mathbf{cdga}_{\mathbb{C}} \longrightarrow \mathbf{SimplSets}$$

$$A \longmapsto \mathrm{Nerve}(\mathbf{Vect}_n^{\mathrm{der}}(X, A))$$

where $\mathbf{Vect}_n^{\mathrm{der}}(X, A)$ is the full sub-category of $\mathbf{Mod}^{\mathrm{der}}(X, A)$ which are *rank n* derived vector bundles on X i.e. $\mathcal{O}_X \otimes A$-dg-modules $\mathcal{M}$ on X which are

- locally on $X_{\mathrm{Zar}} \times A_{\text{ét}}$ equivalent to $(\mathcal{O}_X \otimes A)^n$
- flat over A (more precisely, $\mathcal{M}(U)$ is a cofibrant A-dg-module, for any open $U \subset X$)

DAG - An example: derived stack of vector bundles

Theorem (Toën-V.)

- $\mathbb{R}\mathbf{Vect}_n(X)$ is a derived stack
- If $E \to X$ is a *rk n* vector bundle in X,

$$\mathrm{T}_E(\mathbb{R}\mathbf{Vect}_n(X)) \simeq \mathbb{R}\Gamma_{\mathrm{Zar}}(X_{\mathrm{Zar}}, \mathrm{End}(E))[1]$$

  (the whole complex !)
- $t_0(\mathbb{R}\mathbf{Vect}_n(X)) \simeq \mathbf{Vect}_n(X)$ (the usual underived stack of vector bundles on X)

$\rightsquigarrow$ this is a global realization of Kontsevich hidden smoothness idea.

Obstruction theories in AG

$\mathcal{M}$ - algebraic stack (say over $\mathbb{C}$); $\mathbb{L}_{\mathcal{M}}$ - cotangent complex of $\mathcal{M}$

Obstruction theory for $\mathcal{M}$ (Behrend-Fantechi)

Map $\varphi : \mathbb{E} \to \mathbb{L}_{\mathcal{M}}$ in $D(\mathcal{M})$ such that

- $H^i(\mathbb{E}) = 0$ for $i > 0$, $H^i(\mathbb{E})$ coherent for $i = -1, 0$.
- φ induces an iso on H^0 and surjective on H^{-1}.

If $\mathcal{M}$ is Deligne-Mumford, an obstruction theory is perfect if $\mathbb{E}$ has perfect amplitude in $[-1, 0]$.

- Tangent space (rel to the perfect obstruction theory) $:= H^0(\mathbb{E})$
- Obstruction space (rel to the perfect obstruction theory) $:= H^{-1}(\mathbb{E})$
- virtual dimension of $\mathcal{M}$ (rel to the perfect obstruction theory $\mathbb{E}$) $\mathrm{d}^{\mathrm{vir}}(\mathcal{M}) := \dim H^0(\mathbb{E}) - \dim H^{-1}(\mathbb{E})$.

Rough idea : $\mathcal{M}$ is cut out (locally) by $\dim H^{-1}(\mathbb{E})$ equations in a space of dimension $\dim H^0(\mathbb{E})$.

Obstruction theories in AG

Example - $\overline{\mathcal{M}}_{g,n}(X;\beta)$ - stack of stable maps of type $(g, n; \beta \in H_2(X;\mathbb{Z}))$ to a proj. smooth variety X - has a natural perfect obstruction theory whose virtual dimension

$$\mathrm{d}^{\mathrm{vir}}(\overline{\mathcal{M}}_{g,n}(X;\beta)) = <\beta, c_1(X)> + \dim(X)(1-g) + 3g - 3 + n$$

(Recall - $f : (C; x_1, \ldots, x_n) \to X$ is stable of type $(g, n; \beta)$ if C is proper, reduced, at worst nodal, arithm. genus g curve, $x_1, \ldots, x_n$ distinct and smooth, f has no infinitesimal automorphisms, $f_*[C] = \beta$).

Obstruction theories and virtual fundamental classes

Behrend-Fantechi : perfect obstruction theory on a proper DM stack $\mathcal{M}$ $\Rightarrow$ virtual fundamental class $[\mathcal{M}]^{\mathrm{vir}} \in A_{d^{\mathrm{vir}}}(\mathcal{M};\mathbb{Q})$. Enables definition of enumerative invariants, e.g.

Gromov-Witten invariants

- $\overline{\mathcal{M}}_{g,n}(X;\beta)$ - stack of n-pointed, genus g stable maps to a proj. smooth variety X, hitting $\beta \in H_2(X;\mathbb{Z})$
- $\mathbb{E} \to \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X;\beta)}$ where $\mathbb{E}_f := \mathbb{R}\Gamma(C, \mathrm{Cone}(\mathbb{T}_C(-\sum_i x_i) \to f^*\mathrm{T}_X))^\vee$
- $GW_{g,n}(X,\beta;\gamma_1,\ldots,\gamma_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X;\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*\gamma_1 \cdots \mathrm{ev}_n^*\gamma_n \in A_0(\overline{\mathcal{M}}_{g,n}(X;\beta);\mathbb{Q})$

where:

$-\,\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X;\beta) \to X : (f:(C;x_1,\ldots,x_n) \to X) \mapsto f(x_i)$

$-\,\gamma_i \in A^*(X;\mathbb{Q})$ such that $\sum_i \deg(\gamma_i) = d^{\mathrm{vir}}$

DAG gives natural obstruction theories

A derived DM stack $\mathbb{R}\mathcal{M}$ is quasi-smooth if its cotangent complex $\mathbb{L}_{\mathbb{R}\mathcal{M}}$ is of perfect amplitude in $[-1, 0]$.

Induced obstructions

- $\mathbb{R}\mathcal{M}$ a q-smooth derived DM stack, $i : \mathcal{M} := \mathrm{t}_0(\mathbb{R}\mathcal{M}) \hookrightarrow \mathbb{R}\mathcal{M}$, then $i^*\mathbb{L}_{\mathbb{R}\mathcal{M}} \to \mathbb{L}_{\mathcal{M}}$ is a perfect obstruction theory on $\mathcal{M}$.
- These induced obstruction theories are functorial w.r. to maps of derived stacks (as opposed to the weak functoriality of B-F's).

General expectation: each perfect DM pair $(\mathcal{M}, \mathbb{E} \to \mathbb{L}_{\mathcal{M}})$ comes from a derivation of $\mathcal{M}$ (i.e. a q-smooth derived stack $\mathbb{R}\mathcal{M}$ such that $\mathrm{t}_0(\mathbb{R}\mathcal{M}) \simeq \mathcal{M}$). Verified in all known cases.

So: interesting to consider moduli spaces admitting more than one (geometrically meaningful) obstruction theory.

Obstructions for stable maps to a $K3$

S - smooth projective complex $K3$ surface
$\overline{\mathcal{M}}_{g,n}(S;\beta)$ - DM stack of stable maps of type $(g,n;\beta)$ to S
$(\beta \in H_2(S,\mathbb{Z}) \simeq H^2(S,\mathbb{Z})$ a curve class)

Two obstruction theories

- standard one (existing for any smooth proj X in place of S) $\mathbb{E}_{\mathrm{std}} \to \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(S;\beta)}$ with $\mathbb{E}_{\mathrm{std},f} := \mathbb{R}\Gamma(C, \mathrm{Cone}(\mathbb{T}_C(-\sum_i x_i) \to f^*\mathrm{T}_X))^\vee \rightsquigarrow$ $[\overline{\mathcal{M}}_{g,n}(S;\beta)]^{\mathrm{vir}}_{\mathrm{std}} = 0$ in $A_{g-1+n}(\overline{\mathcal{M}}_{g,n}(S;\beta);\mathbb{Q})$ (hence, trivial GW invariants). ▸ Why?
- Okounkov-Maulik-Pandharipande-Thomas - reduced obstruction theory $\mathbb{E}_{\mathrm{red}} \to \mathbb{L}_{\mathcal{M}_{g,n}(S;\beta)} \rightsquigarrow [\overline{\mathcal{M}}_{g,n}(S;\beta)]^{\mathrm{vir}}_{\mathrm{red}} \neq 0$ in $A_{g+n}(\overline{\mathcal{M}}_{g,n}(S;\beta);\mathbb{Q}) \rightsquigarrow$ nontrivial (reduced) GW invariants.

Problems and how DAG enters

Problems -

- Only the pointwise tgt/obstruction spaces are constructed in literature (but one could fix this...)
- computational, ad-hoc flavor of the construction $\rightsquigarrow$ no clear geometrical interpretation.

Answers

DAG allows for a clear geometrical construction yielding a global (reduced) obstruction theory with the same tgt/obstruction spaces as those of Okounkov-Maulik-Pandharipande-Thomas'.

Derived stack of stable maps $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta)$

Basic lemma - F derived stack, $\mathrm{t}_0 F \hookrightarrow F$ inclusion of the truncation, $U_0 \hookrightarrow \mathrm{t}_0 F$ open substack. Then there is a unique derived open substack $U \hookrightarrow F$ sitting in a homotopy cartesian diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & \mathrm{t}_0 F \\ \downarrow & & \downarrow \\ U & \longrightarrow & F \end{array}$$

We'll use this to define $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta)$.

First step - $\mathbb{R}\mathcal{M}^{\mathrm{pre}}_{g,n}(X) := \mathbb{R}\mathrm{HOM}_{\mathbf{dSt}_{\mathbb{C}}/\mathcal{M}^{\mathrm{pre}}_{g,n}}(\mathcal{C}^{\mathrm{pre}}_{g,n}, X \times \mathcal{M}^{\mathrm{pre}}_{g,n})$,
where $\mathcal{C}^{\mathrm{pre}}_{g,n} \to \mathcal{M}^{\mathrm{pre}}_{g,n}$ - universal family.
Second step - Use $\overline{\mathcal{M}}_{g,n}(X;\beta) \hookrightarrow \overline{\mathcal{M}}_{g,n}(X) \hookrightarrow \mathcal{M}^{\mathrm{pre}}_{g,n}(X)$ (open substacks) and Basic Lemma, to get their derived versions
$\mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta) \hookrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,n}(X) \hookrightarrow \mathbb{R}\mathcal{M}^{\mathrm{pre}}_{g,n}(X)$
with universal family $\mathbb{R}\overline{\mathcal{C}}_{g,n}(X;\beta) \to \mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta) \times X$.

Derived stack of stable maps $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta)$

Properties of the derived stack of stable maps

- $t_0(\mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta)) \simeq \overline{\mathcal{M}}_{g,n}(X;\beta)$
- $t_0(\mathbb{R}\overline{\mathcal{C}}_{g,n}(X;\beta)) \simeq \overline{\mathcal{C}}_{g,n}(X;\beta)$
- the derived tangent complex at $f : (C; x_1, \cdots, x_n) \to X$,

$$\mathbb{T}_{x_f}\mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta) \simeq \mathbb{R}\Gamma(C, \mathrm{Cone}(\mathbb{T}_C(-\sum x_i) \to f^*T_X))$$

- the standard obstruction theory for $\overline{\mathcal{M}}_{g,n}(X;\beta)$ is exactly

$$\mathbb{E}_{\mathrm{std}} = j^*\mathbb{L}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta)} \to \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X;\beta)}$$

where $j : t_0(\mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta)) \simeq \overline{\mathcal{M}}_{g,n}(X;\beta) \hookrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,n}(X;\beta)$.

Reduced derived stack of stable maps to a K3

Main Theorem (Schürg-Toën-V)

Let S be a $K3$ surface. Then $\exists$ a quasi-smooth DM derived stack $\mathbb{R}\overline{\mathcal{M}}^{\mathrm{red}}_{g,n}(S;\beta)$ such that

- $\mathrm{t}_0(\mathbb{R}\overline{\mathcal{M}}^{\mathrm{red}}_{g,n}(S;\beta)) \simeq \overline{\mathcal{M}}_{g,n}(S;\beta)$; hence induces a global $[-1,0]$ perfect obstruction theory

$$\mathbb{E}_{\mathrm{red}} := j^*\mathbb{L}_{\mathbb{R}\overline{\mathcal{M}}^{\mathrm{red}}_{g,n}(S;\beta)} \to \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(S;\beta)}$$

- the pointwise *tangent* spaces $H^0(\mathbb{E}_{\mathrm{red},f})$, and pointwise *obstruction* spaces $H^{-1}(\mathbb{E}_{\mathrm{red},f})$ coincide with those defined by Okounkov-Maulik-Pandharipande-Thomas.

Why usual GW's are trivial for a $K3$?

Short answer - because S is holomorphic symplectic.
Suppose $n = 0$ (unpointed case, for simplicity) take a $\mathbb{C}$-point of $\overline{\mathcal{M}}_{g,n}(S;\beta)$ i.e. a stable map $f : C \to S \rightsquigarrow$ $\mathrm{obs}_f := H^1(C, \mathrm{Cone}(\mathbb{T}_C \to f^* T_S))$ - obstruction space at f sits into ex.seq.

$$H^1(C, \mathbb{T}_C) \to H^1(C, f^* T_S) \to \mathrm{obs}_f \to 0$$

and the composite map (using sympl. form $T_S \simeq \Omega^1_S$)

$$H^1(C, \mathbb{T}_C) \longrightarrow H^1(C, f^* T_S) \simeq H^1(C, f^*\Omega^1_S) \xrightarrow{df} H^1(C, \omega_C) \simeq \mathbb{C}$$

vanishes $\rightsquigarrow$ have an induced trivial 1-dim'l quotient $a : \mathrm{obs}_f \to \mathbb{C}$ which forces $[\overline{\mathcal{M}}_{g,n}(S;\beta)]^{\mathrm{vir}}_{\mathrm{std}} = 0$.
Way out - modify the standard obstruction theory by keeping the same tgt space ($= H^0(C, \mathrm{Cone}(\mathbb{T}_C \to f^* T_S))$) but setting the new obstruction space to ker a.

Back

Quantizing moduli spaces

What follows is joint work with B. Töen, T. Pantev and M. Vaquié.

X - derived stack, $D_{qcoh}(X)$ - dg-category of quasi-coherent complexes on X.
$D_{qcoh}(X)$ is a symmetric monoidal i.e. $E_\infty - \otimes$-dg-category $\Rightarrow$ in particular: a dg-category ($\equiv E_0 - \otimes$-dg-cat), a monoidal dg-category ($\equiv E_1 - \otimes$-dg-cat), a braided monoidal dg-category ($\equiv E_2 - \otimes$-dg-cat), ... $E_n - \otimes$-dg-cat (for any $n \geq 0$).
(Rmk - For ordinary categories $E_n - \otimes \equiv E_3 - \otimes$, for any $n \geq 3$; for ∞-categories, like dg-categories, all different, a priori !)

n-quantization of a derived moduli space

- An *n-quantization* of a derived moduli space X is a (formal) deformation of $D_{qcoh}(X)$ as an $E_n - \otimes$-dg-category.
- Main Theorem - An *n-shifted symplectic form* on X determines an n-quantization of X.

Quantizing moduli spaces

– Main line of the proof –

- Step 1. Show that an n-shifted symplectic form on X induces a n-shifted Poisson structure on X.
- Step 2. A derived extension of Kontsevich formality (plus a fully developed deformation theory for $E_n - \otimes$-dg-category) gives a map

$$\{n\text{-shifted Poisson structures on X}\} \to \{\text{n-quantizations of X}\}.$$

□

We aren't there yet ! We have established Step 2 for all n (using also a recent result by N. Rozenblyum), and Step 1 for X a derived DM stack (all n) ; the Artin case is harder...

Below, I will concentrate on **derived** a.k.a **shifted symplectic structures**.

Derived symplectic structures I - Definition

To generalize the notion of symplectic form in the derived world, we need to generalize the notion of 2-form, of closedness , and of nondegeneracy. In the derived setting, it is closedness the trickier one: it is no more a property but a list of coherent data on the underlying 2-form !

Why? Let A be a (cofibrant) cdga, then $\Omega^{\bullet}_{A/k}$ is a bicomplex : vertical d coming from the differential on A, horizontal d is de Rham differential d_{DR}. So you don't really want $d_{DR}\omega = 0$ but $d_{DR}\omega \sim 0$ with a specified 'homotopy'; but such a homotopy is still a form ω_1

$$d_{DR}\omega = \pm d\omega_1$$

And we further require that $d_{DR}\omega_1 \sim 0$ with a specified homotopy

$$d_{DR}\omega_1 = \pm d(\omega_2),$$

and so on.

This $(\omega, \omega_1, \omega_2, \cdots)$ is an infinite set of higher coherencies data on the underlying form ω, not properties!

Derived symplectic structures I - Definition

More precisely: the guiding paradigm comes from negative cyclic homology: if $X = Spec\, R$ is smooth over k ($\mathrm{char}(k) = 0$) then the HKR theorem tells us that

$$HC_p^-(X/k) = \Omega_{X/k}^{p,cl} \oplus \prod_{i \geq 0} H_{DR}^{p+2i}(X/k)$$

and the summand $\Omega_{X/k}^{p,cl}$ is the weight (grading) p part.

So, a fancy (but homotopy invariant) way of defining classical closed p-forms on X is to say that they are elements in $HC_p^-(X/k)^{(p)}$ (weight p part).

Derived symplectic structures I - Definition

Can use an analog of negative cyclic homology to define

n-shifted (closed) p-forms; derived symplectic forms

X derived Artin stack locally of finite presentation ($\rightsquigarrow \mathbb{L}_X$ is perfect).

- There is a space of n-shifted p-forms on X/k :
  $\mathcal{A}^p(X;n) :=\simeq |\mathbb{R}\Gamma(X, (\wedge^p \mathbb{L}_X)[n])|$. So,
  $\pi_0(\mathcal{A}^p(X;n)) = Hom_{D(X)}(\wedge^p \mathbb{T}_X, \mathcal{O}_X[n])$.
- There is a space of closed n-shifted p-forms on X/k: $\mathcal{A}^{p,cl}(X;n)$
- There is an 'underlying form' map $\mathcal{A}^{p,cl}(X;n) \to \mathcal{A}^p(X;n)$
- Space of n-shifted symplectic forms: $Sympl(X,n) \subset \mathcal{A}^{2,cl}(X;n)$ of non degenerate closed forms (i.e. underlying forms $\omega : \wedge^2 \mathbb{T}_X \to \mathcal{O}_X[n]$ induce $\omega^\flat : \mathbb{T}_X \simeq \mathbb{L}_X[n]$).

Rmks - $|-|$ is the geometric realization; for an n-shifted p-form. Being closed is not a condition, rather: any n-shifted closed p-form has an underlying n-shifted p-form (via the map above); for $n = 0$ and X a smooth underived scheme, we recover the usual notions.

Derived symplectic structures I - Definition

- Nondegeneracy ($\mathbb{T}_X \simeq \mathbb{L}_X[n]$) for X n-shifted symplectic, involves a kind of duality between the stacky (positive degrees) and the derived (negative degrees) parts of $\mathbb{L}_X$
  $\Rightarrow$ X smooth underived scheme may only admit 0-shifted symplectic structures, and these are usual symplectic structures.
- $G = GL_n \Rightarrow BG$ has a canonical 2-shifted symplectic form whose underlying form is defined as follows:
  start from $Sym^2\mathfrak{g} \to k : A \cdot B \to Tr(AB) \Rightarrow Sym^2\mathfrak{g}[2] \simeq \wedge^2 g[1] \to k[2]$, and note that $\mathbb{T}_e BG \simeq \mathfrak{g}[1]$.
- Same as above (with a choice of G-invariant symm bil form on $\mathfrak{g}$) for G reductive over k. Rmk - The induced quantization is the "quantum group" (i.e. quantization is the $\mathbb{C}[[t]]$-braided mon cat given by completion at $q = 1$ of $Rep(G(g))$ $\mathbb{C}[q, q^{-1}]$-braided mon cat).
- As expected, the *n*-shifted cotangent bundle

$$T^*X[n] := Spec_X(Sym(\mathbb{T}_X[-n]))$$

  has a canonical n-shifted symplectic form.

Derived symplectic structures on mapping stacks

There is a unified statement with the following corollary:

Existence Theorem 1 - Derived mapping stacks

Let (F, ω) be n-shifted symplectic derived Artin stack.

- Betti - If $X = M^d$ compact, connected, topological manifold. The choice of fund class $[X]$ yields a canonical $(n-d)$-shifted sympl structure on $MAP(X, F)$.
- Calabi-Yau - X Calabi-Yau smooth and proper k-scheme (or k-DM stack), with geometrically connected fibres of dim d. The choice of a trivialization of the canonical sheaf ω_X yields a canonical $(n-d)$-shifted sympl structure on $MAP(X, F)$.

Example of Betti: X n-symplectic $\Rightarrow$ its derived loop space $LX := MAP(S^1, X)$ is $(n-1)$-symplectic.

Derived symplectic structures on lagrangian intersections

Existence Theorem 2 - Derived lagrangian intersections

Let (F, ω) be n-shifted symplectic derived Artin stack, and $L_i \to F$ a map of derived stacks equipped with a Lagrangian structure, $i = 1, 2$. Then the homotopy fiber product $L_1 \times_F L_2$ is canonically a $(n-1)$-shifted derived Artin stack.
In particular, if $F = Y$ is a smooth symplectic variety, and $L_i \hookrightarrow Y$ is a smooth closed lagrangian subvariety, $i = 1, 2$, then the derived intersection $L_1 \times_F L_2$ is canonically (-1)-shifted symplectic.

Rmk - An interesting case is the derived critical locus $\mathbb{R}Crit(f)$ for f a global function on a smooth symplectic Deligne-Mumford stack Y. Here

$$\begin{array}{ccc} \mathbb{R}Crit(f) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle df} \\ Y & \xrightarrow[0]{} & T^*Y \end{array}$$

Lagrangian intersections: idea of the Proof

(M, ω) smooth symplectic (usual sense); two smooth lagrangians:

$$L_1 \hookrightarrow (M, \omega) \hookleftarrow L_2$$

By definition of derived intersection:

$$L_1 \leftarrow L_1 \times^h_M L_2 \rightarrow L_2$$

$\exists$ canonical homotopy $\omega_1 \sim \omega_2$ between the two pullbacks of ω to $L_{12} := L_1 \times^h_M L_2$.
But L_1, L_2 are lagrangians, so we have an induced self-homotopy $0 \sim 0$ of the zero form on L_{12}.
What is a self-homotopy h of the zero form?
It is a map

$$h : \wedge^2 \mathbb{T}_{L_{12}} \rightarrow \mathcal{O}_{L_{12}}[-1]$$

of complexes (since $hd - dh = 0 - 0 = 0$): so h is a (-1)-shifted 2-form on L_{12}.
Then one checks that such an h actually comes from a closed (-1)-shifted symplectic form on L_{12}. $\square$

Derived symplectic structure on $\mathbb{R}\mathbf{Perf}$

Consider

$$\mathbb{R}\mathbf{Perf} : \mathbf{cdga}_k^{\leq 0} \to SSets \ : A \mapsto Nerve(Perf(A)^{cof}, q-iso)$$

where $Perf(A)$ is the subcategory of all A-dg-modules consisting of dualizable objects (= homotopically finitely presented = compact objects in $D(A)$).

The tangent complex at $E \in \mathbb{R}\mathbf{Perf}(k)$ is $\mathbb{T}_E \mathbb{R}\mathbf{Perf} \simeq \mathbb{R}End(E)[1]$.

$\mathbb{R}\mathbf{Perf}$ is locally Artin of finite presentation.

Existence theorem 3 - $\mathbb{R}\mathbf{Perf}$ is 2-shifted symplectic

The derived stack $\mathbb{R}\mathbf{Perf}$ is 2-shifted symplectic.

Derived symplectic structure on $\mathbb{R}\mathbf{Perf}$

Corollary of thms 1 (MAP) and 3 (RPerf)

X Calabi-Yau smooth and proper k-scheme (or k-DM stack), with geometrically connected fibres of dim d. The choice of a trivialization of the canonical sheaf ω_X yields a canonical $(2-d)$-shifted sympl structure on $MAP(X, \mathbb{R}\mathbf{Perf}) = \mathbb{R}\mathbf{Perf}(X)$.

In particular, if X is a CY 3-fold, $\mathbb{R}\mathbf{Perf}(X)$ is (-1)-shifted symplectic. As a corollary, one gets a solution to a longstanding problem in Donaldson-Thomas theory:

Corollary (Brav-Bussi-Joyce, 2013)

The Donaldson-Thomas moduli space of simple perfect complexes (with fixed determinant) on a Calabi-Yau 3-fold is locally for the Zariski topology the critical locus of a function, the *DT-potential* on a smooth complex manifold. Locally the obstruction theory on the DT moduli space is given by the (-1)-symplectic form on the derived critical locus of the potential.

Thank you!