A construction of 2-parameter family of Painlevé τ-functions via the topological recursion

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Overview

- Review of Painlevé equations (with a formal parameter $\hbar$)

$$(P_{\mathrm{I}}) \quad : \quad \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t \quad \text{(the first Painlevé equation)}$$

- Review of Eynard-Orantin's topological recursion

$$\text{spectral curve} \; \text{—} \boxed{\text{topological recursion}} \rightsquigarrow W_{g,n}(z_1, \ldots, z_n), \; F_g$$

(and a construction of "0-parameter solutoin" of Painlevé equation)

- **Main Result** : Construction of "2-parameter solution" of Painlevé equation

Theorem ([I, in preparation])

Let $Z(t, \hbar; \nu) := \exp(\sum_{g \geq 0} \hbar^{2g-2} F_g(t, \nu))$ be the topological recursion partition function of the spectral curve $y^2 = 4x^3 + 2tx + u(t, \nu)$. Then, the Fourier series

$$\tau_{P_{\mathrm{I}}}(t, \hbar; \nu, \rho) = \sum_{k \in \mathbb{Z}} e^{2\pi i k \rho / \hbar} \cdot Z(t, \hbar; \nu + k\hbar)$$

gives a 2-parameter family (parametrized by ν, ρ) of (formal) τ-function for (P_{I}). ($W_{g,n}$ gives the isomonodromic wave function associated with (P_{I}).)

Painlevé equations

Painlevé Equations (P_J) $(J = \mathrm{I}, \ldots, \mathrm{VI})$

- **Painlevé equations** are non-linear ODEs discovered by Painlevé-Gambier:

$$(P_{\mathrm{I}}) \quad : \quad \frac{d^2q}{dt^2} = 6q^2 + t.$$

$$(P_{\mathrm{III}}) \quad : \quad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \left(\frac{q^3}{t^2} - \frac{\theta_\infty q^2}{t^2} + \frac{\theta_0}{t} - \frac{1}{q}\right).$$

$$(P_{\mathrm{VI}}) \quad : \quad \frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt}$$
$$+ \frac{q(q-1)}{2t(t-1)(q-t)} + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{\theta_\infty^2}{4} - \frac{\theta_0^2}{4}\frac{t}{q^2} + \frac{\theta_1^2}{4}\frac{t-1}{(q-1)^2} - \frac{\theta_t^2}{4}\frac{t(t-1)}{(q-t)^2}\right).$$

- **Painlevé property**: Movable singularities must be poles.
- **Many nice properties**:
  Isomonodromy deformation (**integrability**), Hamiltonian description, affine-Weyl symmetry, space of initial conditions, transcendency of solutions, connection formula, conformal block expansion of solutions,...
- **Relation to other topics**:
  Ising model, random matrix theory, Frobenius-Saito structure, tt^*-geometry, conformal field theory, gauge theory...

Isomonodoromic Deformation (Integrability)

Fact : $(P_J) \Leftrightarrow$ compatibility condition of system of linear PDEs

For example, (P_{I}) is equivalent to the compatibility condition ($[L, M] = 0$) of

$$(L_{\mathrm{I}}) \;:\; L\Psi := \left[\frac{\partial^2}{\partial x^2} - \frac{1}{x-q}\left(\frac{\partial}{\partial x} - p\right) - (4x^3 + 2tx + 2H)\right]\Psi = 0$$

$$(D_{\mathrm{I}}) \;:\; M\Psi := \left[\frac{\partial}{\partial t} - \frac{1}{2(x-q)}\left(\frac{\partial}{\partial x} - p\right)\right]\Psi = 0$$

with

$$p = \frac{dq}{dt} \quad \text{and} \quad H = \frac{p^2}{2} - 2q^3 - tq$$

- $(P_{\mathrm{I}}) : \dfrac{d^2q}{dt^2} = 6q^2 + t \quad \Leftrightarrow \quad$ Hamiltonian system $\dfrac{dq}{dt} = \dfrac{\partial H}{\partial p}, \; \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q}$
- **Monodromy / Stokes matrices for $L\Psi = 0$ is independent of t.**
  ($\rightsquigarrow$ Monodromy / Stokes data are first integrals.)

τ-function (for Painlevé I)

$$(P_{\mathrm{I}}) \ : \ \frac{d^2 q}{dt^2} = 6q^2 + t, \quad p = \frac{dq}{dt}, \quad H = \frac{p^2}{2} - 2q^3 - tq$$

- The **τ-function** is defined (up to constant) by

$$\frac{d}{dt} \log \tau_{P_{\mathrm{I}}} = H \quad \left(\text{or} \ \frac{d^2}{dt^2} \log \tau_{P_{\mathrm{I}}} = -q\right)$$

  $\tau_{P_{\mathrm{I}}}$ is entire (although q and p have poles). [Okamoto, Jimbo-Miwa-Ueno]
- Analogy to Weierstrass functions :

$$q \leftrightarrow \wp, \qquad H \leftrightarrow \zeta, \qquad \tau_{P_{\mathrm{I}}} \leftrightarrow \sigma \ \text{(or } \theta\text{-function)}$$

- τ-function satisfies Hirota-type bilinear equation :

$$D^4 \tau_{P_{\mathrm{I}}} \cdot \tau_{P_{\mathrm{I}}} + 2t \, \tau_{P_{\mathrm{I}}} \cdot \tau_{P_{\mathrm{I}}} = 0$$

  (where $D^n f \cdot g = \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot \frac{d^k f}{dt^k} \cdot \frac{d^{n-k} g}{dt^{n-k}}$ is the Hirota derivative.)

General Solution of Painlevé VI via CFT

Theorem ([Gamayun-Iorgov-Lisovyy 12], ...)

The following Fourier series gives 2-parameter family of τ-functions for (P_{VI}):

$$\tau_{P_{\mathrm{VI}}}(t;\nu,\rho) = \sum_{k\in\mathbb{Z}} e^{2\pi i k\rho} \cdot C(\nu+k) \cdot t^{(\nu+k)^2-\theta_0^2-\theta_t^2} \cdot B(t;\nu+k)$$

where $B(t,\nu)$ is 4-point Virasoro conformal block (with $c=1$).

- By AGT relation, $B(t;\nu) = (1-t)^{2\theta_t\theta_1} \sum_{\lambda,\nu\in\mathbb{Y}} B_{\lambda,\mu}(\nu) t^{|\lambda|+|\mu|}$ with

$$B_{\lambda,\mu}(\nu) = \prod_{(i,j)\in\lambda} \frac{((\theta_t+\nu+i-j)^2-\theta_0^2)\cdot((\theta_1+\nu+i-j)^2-\theta_\infty^2)}{h_\lambda^2(i,j)\cdot(\lambda'_j+\mu_i-i-j+1+2\nu)^2} \times \text{(similar factor for } \mu)$$

- The constant factor C is expressed by Barnes G-functions:

$$C(\nu) = \frac{\prod_{\epsilon,\epsilon'\in\{\pm1\}} G(1+\theta_t+\epsilon\theta_0+\epsilon'\nu)\cdot G(1+\theta_1+\epsilon\theta_\infty+\epsilon'\nu)}{\prod_{\epsilon\in\{\pm1\}} G(1+2\epsilon\nu)}$$

- ν and ρ are integration constants (general solution !)
  Monodromy data is parametrized by ν, ρ (and θ_*) explicitly.
- Similar formula is proved/conjectured for other (q-)Painlevé equations ([Nagoya, Bonelli et.al,...]). **No explicit formula is known for (P_{I}).**

WKB Theoretic Approach

- Introduce a small parameter $\hbar$: $t \mapsto \hbar^{-4/5} t,\ q \mapsto \hbar^{-2/5} q$ yields

$$(P_{\rm I}) \ \mapsto \ \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t$$

(cf. [Kawai-Takei 96 ∼])

- **WKB-type formal solutions of** $(P_{\rm I})$:
  - **0-parameter solution** = the formal power series solution.

$$q = q(t,\hbar) = q_0(t) + \hbar q_1(t) + \hbar^2 q_2(t) + \cdots$$

Leading term satisfies $6q_0^2 + t = 0$.

  - **1-parameter solution** = trans-series solution.
  - **2-parameter solution** [Aoki-Kawai-Takei 96]:

$$\begin{aligned} q(t,\hbar;\alpha,\beta) = q_0(t) &+ \hbar^{\frac{1}{2}} \left(\alpha\, a_{+1}(t)\, e^{+\phi(t)/\hbar} + \beta\, a_{-1}(t)\, e^{-\phi(t)/\hbar} \right) \\ &+ \hbar \left(\alpha^2\, a_{+2}(t)\, e^{+2\phi(t)/\hbar} + \alpha\beta\, a_0(t) + \beta^2\, a_{-2}(t)\, e^{-2\phi(t)/\hbar} \right) \\ &+ \cdots \end{aligned}$$

where $\phi(t) = \mathrm{const} \times t^{5/4}$.

Question : Relation to conformal block solutions ?

WKB Theoretic Approach (cont.)

- Corresponding isomonodromy system is also rescaled

$$(L_{\rm I}) \;\mapsto\; L\Psi := \left[\hbar^2 \frac{\partial^2}{\partial x^2} - \frac{\hbar}{x-q}\left(\hbar\frac{\partial}{\partial x} - p\right) - (4x^3 + 2tx + 2H)\right]\Psi = 0$$

$$(D_{\rm I}) \;\mapsto\; M\Psi := \left[\hbar \frac{\partial}{\partial t} - \frac{1}{2(x-q)}\left(\hbar\frac{\partial}{\partial x} - p\right)\right]\Psi = 0$$

with

$$p = \hbar\frac{dq}{dt} \quad \text{and} \quad H = \frac{p^2}{2} - 2q^3 - tq$$

- The τ-function is defined by $\hbar^2 \dfrac{d}{dt}\log\tau_{P_{\rm I}} = H$.

- We will apply the **topological recursion** (and the theory of **quantum curves**) to the "classical limit" of $(L_{\rm I})$:

$$y^2 = 4x^3 + 2tx + 2H_0$$

Topological Recursion

Spectral Curve

Definition

A **spectral curve** is a triplet (Σ, x, y), where

- Σ : compact Riemann surface with a prescribed symplectic basis of $H_1(\Sigma; \mathbb{Z})$.
- x, y : meromorphic functions on Σ.

such that dx and dy never vanish simultaneously.

- Example 1 (Airy curve) :

$$\Sigma = \mathbb{P}^1, \quad x(z) = z^2, \quad y(z) = z. \qquad (y^2 - x = 0)$$

- Example 2 (Elliptic curve) :

$$\Sigma = \mathbb{C}/\Lambda, \quad x(z) = \wp(z), \quad y(z) = \wp'(z). \qquad (y^2 = 4x^3 - g_2 x - g_3)$$

where $\Lambda = \mathbb{Z}\omega_A + \mathbb{Z}\omega_B$ (with $\mathbb{R}$-independent complex numbers ω_A, ω_B), and

$$\wp(z) = \wp(z; \omega_A, \omega_B) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

$$g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6}$$

Eynard-Orantin Correlators

Definition [Eynard-Orantin 07]

To a given spectral curve (Σ, x, y), define

$$\{W_{g,n}(z_1, \ldots, z_n)\}_{g\geq 0, n\geq 1} \ : \ \text{a sequence of meromorphic multi-differentials on } \Sigma$$

by the following recursion relation (called **topological recursion**):

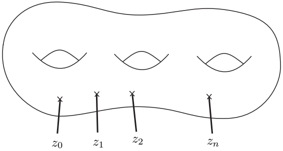
$$W_{0,1}(z) := y(z)dx(z), \quad W_{0,2}(z_1, z_2) := \text{Bergman bi-differential}$$

$$W_{g,n+1}(z_0, z_1, \ldots, z_n) := \sum_{a \,:\, \text{ramification point}} \operatorname*{Res}_{z=a} K_a(z_0, z) \Bigg(W_{g-1,n+2}(z, \bar{z}, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g,\ I_1 \sqcup I_2 = \{1,\ldots,n\}, \\ \text{except for } (g_i=0\ \&\ I_i=\emptyset)}} W_{g_1,1+|I_1|}(z, z_{I_1})\, W_{g_2,1+|I_2|}(\bar{z}, z_{I_2}) \Bigg).$$

- $W_{0,2}(z_1, z_2) = \dfrac{dz_1 dz_2}{(z_1 - z_2)^2}$ (for $\Sigma = \mathbb{P}^1$), and $\left(\wp(z_1 - z_2) + \dfrac{\eta_A}{\omega_A}\right) dz_1 dz_2$ (for elliptic curve).
- Ramification points are zeros of dx (assume that they are simple).
- $\bar{z}$ is the local conjugation of z near a ramification point.
- $K_a(z_0, z) := \dfrac{1}{2(y(z) - y(\bar{z}))\, dx(z)} \displaystyle\int_{w=\bar{z}}^{w=z} W_{0,2}(z_0, w)$ is the **recursion kernel**.

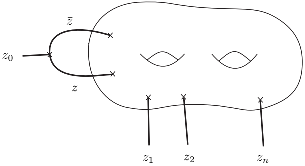
Diagrammatic Expression of Topological Recursion

"$W_{g,n}(z_1, \ldots, z_n)$ $\longleftrightarrow$ genus g Riemann surface with n marked points"

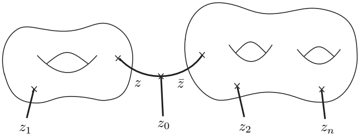


$W_{g,n+1}(z_0, z_1, \ldots, z_n)$

"Degeneration" of Riemann surfaces



$K_a(z_0, z)W_{g-1,n+2}(z, \bar{z}, z_1, \ldots, z_n)$



$K_a(z_0, z)W_{g_1,1+|I_1|}(z, z_{I_1})W_{g_2,1+|I_2|}(\bar{z}, z_{I_2})$

Free Energy and Partition Function

Definition [Eynard-Orantin 07]

- For $g \geq 2$, define **g-th free energy F_g** of the spectral curve by

$$F_g := \frac{1}{2-2g} \sum_{a\,:\,\text{ramification points}} \operatorname*{Res}_{z=a} \Phi(z) W_{g,1}(z) \qquad \left(\Phi(z) := \int^z y(z)dx(z)\right)$$

  (F_0 and F_1 are also defined but in a different manner.)

- **Free energy F** and **partition function Z** of the spectral curve are defined by

$$F = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g$$

$$Z = \exp(F) = \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} F_g\right)$$

Properties [Eynard-Orantin 07]

- $W_{g,n}(z_1, \ldots, z_n)$: holomorphic (as a differential of each z_i) on $\Sigma \setminus R$.
- $W_{g,n}(\cdots, z_i, \cdots, z_j, \cdots) = W_{g,n}(\cdots, z_j, \cdots, z_i, \cdots)$.
- $W_{g,n}$ is normalized along A-cycles $A_1, \ldots, A_{g(\Sigma)}$:

$$\oint_{z_1 \in A_j} W_{g,n}(z_1, \ldots, z_n) = 0 \quad \text{except for } (g,n) = (0,1)$$

- $W_{g,n}$ satisfies differentiation formulas (with respect to moduli parameters). For example, the differentiation with respect to

$$\nu_j := \frac{1}{2\pi i} \oint_{A_j} W_{0,1}(z) \quad (j = 1, \ldots, g(\Sigma))$$

is given by

$$\frac{\partial}{\partial \nu_j} W_{g,n}(z(x_1), \ldots, z(x_n)) = \oint_{x_{n+1} \in B_j} W_{g,n+1}(z(x_1), \ldots, z(x_n), z(x_{n+1}))$$

$$\frac{\partial}{\partial \nu_j} F_g = \oint_{z \in B_j} W_{g,1}(z)$$

TR and Various Geometric Invariants

- Airy curve $(\mathbb{P}^1, x(z) = z^2, y(z) = z)$
  $\rightsquigarrow$ Gromov-Witten invariants for point:

$$W^{\text{Airy}}_{g,n}(z_1, \ldots, z_n) = \frac{1}{2^{2g-2+n}} \sum_{d_1,\ldots,d_n \geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \cdot \prod_{i=1}^{n} \frac{(2d_i - 1)!!}{z_i^{2d_i}} dz_i$$

- Landau-Ginzburg mirror of $\mathbb{P}^1$ $(\mathbb{C}^*, x(z) = z + z^{-1}, y(z) = \log z)$
  $\rightsquigarrow$ Gromov-Witten invariants for $\mathbb{P}^1$.
  [Norbury-Scott 14], [Dunin-Barkowski et.al 13], [Fang et.al 16].
- Bouchard-Klemm-Mariño-Pasquetti conjecture on open Gromov-Witten invariants for toric CY3. [Bouchard et.al 08], [Eynard-Orantin 13]
- KdV τ-function [Kontsevich 92], [Eynard-Orantin 07],
- Painlevé τ-functions (corresponding to “0-parameter solution”) [Borot-Eynard 09, I-Saenz 15, I-Marchal-Saenz 17].
- $\cdots$

Topological Recursion and WKB : Quantum Curve

$$(\mathbb{P}^1, x(z) = z^2, y(z) = z) \quad : \quad \text{Airy curve } (y^2 = x)$$

$$W_{0,1}(z_1) = y(z)dx(z) = 2z_1^2 dz_1, \quad W_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

$$W_{0,3}(z_1, z_2, z_3) = -\frac{dz_1 dz_2 dz_3}{2z_1^2 z_2^2 z_3^2}, \quad W_{1,1}(z_1) = -\frac{dz_1}{16 z_1^4}, \quad \ldots$$

Theorem [Gukov-Sułkowski 12, Dumitrescu-Mulase 14, ...]

The formal series

$$\psi(x, \hbar) = \exp\left(\sum_{g \ge 0, n \ge 1} \frac{\hbar^{2g-2+n}}{n!} \int_\infty^{z(x)} \cdots \int_\infty^{z(x)} W_{g,n}(z_1, \ldots, z_n) \right)$$

is a WKB solution of

$$\left(\hbar^2 \frac{d^2}{dx^2} - x \right) \psi(x, \hbar) = 0$$

(Precisely speaking, we need to regularize the term corresponding to $(g, n) = (0, 2)$ since $W_{0,2}(z_1, z_2)$ has singularity along $z_1 = z_2$.)

Construction of 0-parameter Painlevé τ-function

$$(P_{\rm I}) \ : \ \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t$$

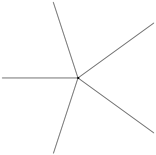
- **0-parameter solution** = the formal power series solution of $(P_{\rm I})$:

$$q = q(t,\hbar) = q_0(t) + \hbar q_1(t) + \hbar^2 q_2(t) + \cdots$$

  Leading term must satisfy $6q_0^2 + t = 0$.
- The corresponding (formal) τ-function has an $\hbar$-expansion:

$$\tau_{P_{\rm I}}(t,\hbar) = \exp\left(\sum_{g \ge 0} \hbar^{2g-2} \tau_g(t) \right)$$

- These formal series are divergent but Borel summable if $\arg t \neq \pi + \frac{2m\pi}{5}$ $(m = 0, \pm 1, \pm 2, \ldots)$.



Construction of 0-parameter Painlevé τ-function (cont.)

- The classical limit of $(L_{\rm I})$ corresponding to 0-parameter solution is

$$y^2 = 4x^3 + 2tx - 4q_0^3 - 2tq_0 = 4(x-q_0)^2(x+2q_0)$$

$\rightsquigarrow$ regard this as a spectral curve $(\mathbb{P}^1, x(z) = z^2 - 2q_0, y(z) = 2z(z^2 - 3q_0))$.

Theorem [I-Saenz 2015]

Let $W_{g,n}$ and F_g be the Eynard-Orantin correlator and free energy of the above spectral curve. Then,

- The formal series

$$\tau_{P_{\rm I}}(t,\hbar) = Z(t,\hbar) = \exp\left(\sum_{g\ge 0} \hbar^{2g-2} F_g(t)\right)$$

is the τ-function corresponding to 0-parameter solution of $(P_{\rm I})$.

- The WKB-type formal series

$$\psi(x,t,\hbar) = \exp\left(\sum_{g\ge 0, n\ge 1} \frac{\hbar^{2g-2+n}}{n!} \int_\infty^{z(x)} \cdots \int_\infty^{z(x)} W_{g,n}(z_1,\dots,z_n)\right)$$

satisfies the isomonodoromy system $(L_{\rm I})$ and $(D_{\rm I})$ associated with $(P_{\rm I})$.

Main Result

A Family of Genus 1 Spectral Curves

Consider a family of elliptic curves

$$y^2 = 4x^3 + 2tx + u(t, \nu)$$

(with a prescribed A-cycle and B-cycle such that $\mathrm{Im}(\omega_B/\omega_A) > 0$) satisfying

$$\nu := \frac{1}{2\pi i} \oint_A y dx \quad \text{is **independent of** } t.$$

- The condition requires

$$\frac{\partial u}{\partial t} = 2\frac{\eta_A}{\omega_A} \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \frac{4\pi i}{\omega_A}$$

- Regard this as a spectral curve

$$\Sigma = \mathbb{C}/\Lambda, \quad x(z) = \wp(z), \quad y(z) = \wp'(z)$$

by the Weierstrass $\wp$-function. $\rightsquigarrow W_{g,n}(z_1, \ldots, z_n)$ and F_g by TR.

Lemma (cf. [Eynard-Orantin 07])

$$\frac{\partial F_0}{\partial t} = \frac{1}{2}u, \quad \frac{\partial F_0}{\partial \nu} = \oint_B y dx, \quad \frac{\partial^2 F_0}{\partial \nu^2} = 2\pi i \frac{\omega_B}{\omega_A}$$

(i.e., F_0 is the Seiberg-Witten prepotential.)

Key Lemma (Quantization of Spectral Curve)

Lemma

The WKB-type formal series

$$\psi_{\pm}(x,t,\hbar;\nu) = \exp\left(\sum_{g\geq 0, n\geq 1} \frac{(\pm\hbar)^{2g-2+n}}{n!} \int_0^{z(x)} \cdots \int_0^{z(x)} W_{g,n}(z_1,\ldots,z_n) \right)$$

satisfies

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - 2\hbar^2 \frac{\partial}{\partial t} - \left(4x^3 + 2tx + 2\hbar^2 \frac{\partial}{\partial t} F(t,\hbar;\nu) \right) \right] \psi_{\pm} = 0$$

- The above PDE is a quantization of $y^2 = 4x^3 + 2tx + u(t,\nu)$.
- **Formal monodoromy relation** along A-cycle and B-cycle:

$$\psi_{\pm}(x,t,\hbar,\nu) \mapsto \begin{cases} e^{\pm 2\pi i \nu/\hbar} \cdot \psi_{\pm}(x,t,\hbar;\nu) & \text{along } A\text{-cycle} \\ \dfrac{Z(t,\hbar,\nu\pm\hbar)}{Z(t,\hbar,\nu)} \cdot \psi_{\pm}(x,t,\hbar;\nu\pm\hbar) & \text{along } B\text{-cycle} \end{cases}$$

Here Z is the TR partition function:

$$Z(t,\hbar;\nu) = \exp(F(t,\hbar;\nu)) = \exp\left(\sum_{g\geq 0} \hbar^{2g-2} F_g(t,\nu) \right)$$

Main Theorem

The formal monodromy relations for $\psi_\pm$ imply that the Fourier series

$$\tilde{\Psi}_\pm(x,t,\hbar;\nu,\rho) := \sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} \cdot Z(t,\hbar;\nu+k\hbar) \cdot \psi_\pm(x,t,\hbar;\nu+k\hbar)$$

has t**-independent formal monodromy**:

$$\tilde{\Psi}_\pm(x,t,\hbar;\nu,\rho) \mapsto \begin{cases} e^{\pm 2\pi i\nu/\hbar} \cdot \tilde{\Psi}_\pm(x,t,\hbar;\nu,\rho) & \text{along } A\text{-cycle} \\ e^{\mp 2\pi i\rho/\hbar} \cdot \tilde{\Psi}_\pm(x,t,\hbar;\nu,\rho) & \text{along } B\text{-cycle} \end{cases}$$

Theorem [I, in preparation]

The formal series

$$\Psi_\pm(x,t,\hbar;\nu,\rho) := \frac{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} \cdot Z(t,\hbar;\nu+k\hbar) \cdot \psi_\pm(x,t,\hbar;\nu+k\hbar)}{\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} \cdot Z(t,\hbar;\nu+k\hbar)}$$

is a **WKB-type solution** of the isomonodoromy system $(L_{\rm I})$ and $(D_{\rm I})$ associated with $(P_{\rm I})$. Here H, q, p in the isomonodromy system are given by

$$H = \hbar^2 \frac{d}{dt} \log\left(\sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar} \cdot Z(t,\hbar;\nu+k\hbar)\right), \quad q = -\hbar\frac{dH}{dt}, \quad p = \hbar\frac{dq}{dt}.$$

Main Theorem (cont)

Theorem [I, in preparation]

The formal series

$$\tau_{P_{\mathrm{I}}}(t,\hbar;\nu,\rho) := \sum_{k\in\mathbb{Z}} e^{2\pi i k\rho/\hbar}\cdot Z(t,\hbar;\nu+k\hbar)$$

is a **2-parameter (formal) τ-function** for (P_{I}).

Remark : [Eynard-Mariño 09], [Borot-Eynard 12] observed that the above Fourier series can be expressed as a formal power series of $\hbar$ whose coefficients are described by θ-functions (and their derivatives) :

$$\tau_{P_{\mathrm{I}}} = Z(\nu)\cdot\left[\theta(z,\tau)+\hbar\left(\frac{1}{6}\frac{\partial^3 F_0}{\partial\nu^2}\theta'''(z,\tau)+\frac{\partial F_1}{\partial\nu}\theta'(z,\tau)\right)+\cdots\right]_{z=\frac{\phi(t)+\rho}{\hbar},\,\tau=\frac{\omega_B}{\omega_A}}$$

where

$$\theta(z,\tau)=\sum_{k\in\mathbb{Z}} e^{2\pi i k z+\pi i k^2\tau}$$

and

$$\phi(t)=\frac{1}{2\pi i}\oint_B y\,dx=\frac{1}{2\pi i}\frac{\partial F_0}{\partial\nu}.$$

(Borel summability is not known...)

Problems and Questions

- Generalization to other Painlevé equations ?
  (cf. [Bonelli et.al 15], [Grassi-Gu 18], [Coman-Pomoni-Teschner 18], ...)
- Explicit and combinatorial expression for the expansion of τ-function ?
  (In terms of Barnes G-function)
- Analytic results ?
  (Borel summability and resurgence property of the formal series.
  Non-linear Stokes phenomenon.)
- Direct monodormy problem / Riemann-Hilbert problem ?
  (Exact WKB theoretic computation of actual monodormy.
  Relation to Gaiotto-Moore-Neitzke's spectral networks.)
- Relation to irregular conformal blocks ?
- Relation to cluster algebras, Bridgeland stability, wall-crossing formulas,....?

Thank you for your attention !