A family of flat connections on the projective space having dihedral monodromy and algebraic Garnier solutions

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Garnier system

The N-variable Garnier system $\mathcal{G}_N$

$$\mathcal{G}_N\colon \begin{cases} \dfrac{\partial \rho_j}{\partial t_i} = -\dfrac{\partial K_i}{\partial \nu_j} & i,j=1,\dots,N \\ \dfrac{\partial \nu_j}{\partial t_i} = \dfrac{\partial K_i}{\partial \rho_j} & i,j=1,\dots,N, \end{cases}$$

where K_i is defined by

$$-\frac{\Lambda(t_i)}{T'(t_i)}\left[\sum_{k=1}^{N}\frac{T(\nu_k)}{(\nu_k-t_i)\Lambda'(\nu_k)}\left\{\rho_k^2-\sum_{m=1}^{N+2}\frac{\theta_m-\delta_{im}}{\nu_k-t_m}\rho_k+\frac{\kappa}{\nu_k(\nu_k-1)}\right\}\right]$$

with $t_{N+1}=0$, $t_{N+2}=1$, $\kappa := \frac{1}{4}\left\{(\sum_{m=1}^{N+2}\theta_m-1)^2-(\theta_\infty^2+1)\right\}$, $\Lambda(t) := \prod_{k=1}^{N}(t-\nu_k)$ and $T(t) := \prod_{k=1}^{N+2}(t-t_k)$. Here θ_m ($m=1,\dots,N+2,\infty$) is the constant parameters.

- Let $\mathcal{A}(x)$ be the Fuchsian system with $N+3$ regular singularities at $t_1,\dots,t_{N+2},\infty$:

$$\mathcal{A}(x) = d + \sum_{i=1}^{N+2} A_i\frac{dx}{x-t_i},$$

where A_i ($i=1,\dots,2n$) are 2×2 matrices independent of x and $t_i \neq t_j$ ($i\neq j$). We assume that $A_\infty := -\sum_{i=1}^{N+2}A_i$ is a diagonal matrix and the eigenvalues of A_i ($i=1,\dots,N+2,\infty$) are $\frac{\theta_i}{2}$ and $-\frac{\theta_i}{2}$.

- Moduli space

$$\mathcal{M}_{\boldsymbol{\theta}} := \left\{(A_1,\dots,A_{N+2},A_\infty)\ \middle|\ \begin{array}{l} A_1+\cdots+A_{N+2}+A_\infty=0 \\ A_i \sim \begin{pmatrix}\frac{\theta_i}{2} & 0 \\ 0 & -\frac{\theta_i}{2}\end{pmatrix}\end{array}\right\} // \mathrm{SL}_2(\mathbb{C})$$

- Put $t_{N+1}=0$, $t_{N+2}=1$.

$$T := \left\{(t_1,\dots,t_N)\ \middle|\ \begin{array}{l} t_i\neq t_j\ (i\neq j) \\ t_i \neq 0, t_i\neq 1\end{array}\right\}.$$

Let U be a open set of T.

- Let σ be a section of the projection $\mathcal{M}_{\boldsymbol{\theta}}\times U \to U$:

$$\sigma\colon \boldsymbol{t}=(t_1,\dots,t_N)\longmapsto d+\sum_{i=1}^{N+2}A_i(\boldsymbol{t})\frac{dx}{x-t_i}\in\mathcal{M}_{\boldsymbol{\theta}}\times U$$

Monodromy representation:

$$d+\sum_{i=1}^{N+2}A_i(\boldsymbol{t})\frac{dx}{x-t_i}\longmapsto \rho(\boldsymbol{t})\colon \pi_1(\mathbb{P}^1\setminus\{0,1,t_1,\cdots,t_N,\infty\})\to\mathrm{SL}_2(\mathbb{C}).$$

ρ is *isomonodromic* if the conjugacy class $[\rho(\boldsymbol{t})]$ is independent of $\boldsymbol{t}$.

Proposition (well-known)

Let $\{\nu_1,\dots,\nu_N\}$ be the roots of the following equation of degree N:

$$\sum_{k=1}^{N+2}\frac{(A_k)_{12}}{x-t_k}=0.$$

For each ν_i, we define ρ_i by

$$\rho_i := \sum_{k=1}^{N+2}\frac{(A_k)_{11}+\frac{\theta_k}{2}}{\nu_i-t_k}.$$

If a tuple $(A_i(\boldsymbol{t}))_{i=1,\dots,N+2}$ is an isomonodromic family, then the corresponding functions $\nu_j(t_1,\dots,t_N)$ and $\rho_j(t_1,\dots,t_N)$ ($j=1,\dots,N$) satisfy the Garnier system $\mathcal{G}_N$.

Main Result

- Let $[x:y:z_1:\dots:z_{n-2}:t]$ be the homogeneous coordinates of $\mathbb{P}^n$.

$$\begin{aligned} f(x,y,t) &:= x^2+y^2+t^2-2(xy+yt+tx) \\ \mathcal{Q}_0 &:= (f(x,y,t)=0) \\ \mathcal{Q}_i &:= (f(x,y,t)-z_i^2=0), \quad i=1,\dots,n-2 \\ D_n &:= (x=0)+(y=0)+(t=0)+\mathcal{Q}_0+\mathcal{Q}_1+\cdots+\mathcal{Q}_{n-2}. \end{aligned}$$

- Let $\boldsymbol{D}_\infty$ be the infinite dihedral group:

$$\boldsymbol{D}_\infty := \left\langle \begin{pmatrix}0 & \alpha \\ -\alpha^{-1} & 0\end{pmatrix}, \begin{pmatrix}\beta & 0 \\ 0 & \beta^{-1}\end{pmatrix}\ \middle|\ \alpha,\beta\in\mathbb{C}^*\right\rangle \leq \mathrm{SL}_2(\mathbb{C}).$$

- For $\boldsymbol{\lambda}=(\lambda_0,\dots,\lambda_{n-1})\in\mathbb{C}^n$, we define rational 1-forms on $\mathbb{P}^n$ as follows:

$$\alpha_0(x,y) := -\frac{(2\lambda_0+\lambda_1)dx-(2\lambda_1+\lambda_0)dy}{2}-\frac{\lambda_1(y-1)}{2}\frac{dx}{x}+\frac{\lambda_0(x-1)}{2}\frac{dy}{y},$$

$$\begin{aligned} \alpha_1(x,y) &:= -\frac{1}{4}\frac{df(x,y,1)}{f(x,y,1)}, \\ \alpha_2(x,y) &:= -\frac{\alpha_0(x,y)}{f(x,y,1)}, \\ \alpha_0^i(x,y,z_i) &:= \lambda_{i+1}\left(dz_i-\frac{z_i d(f(x,y,1)-z_i^2)}{2(f(x,y,1)-z_i^2)}\right), \\ \alpha_2^i(x,y,z_i) &:= -\frac{\alpha_0^i(x,y,z_i)}{f(x,y,1)}, \end{aligned}$$

which are described by the affine coordinates $[x:y:z_1:\dots:z_{n-2}:1]$.

Main result ([1] for $n=2$, [2] for $n\geq 3$)

For generic $\boldsymbol{\lambda}=(\lambda_0,\dots,\lambda_{n-1})\in\mathbb{C}^n$, there exists an explicit flat meromorphic $\mathfrak{sl}_2$-connection $\nabla_{\boldsymbol{\lambda}}$ over the trivial vector bundle $\mathbb{P}^n\times\mathbb{C}^2\to\mathbb{P}^n$ with the following properties:

(i) $\nabla_{\boldsymbol{\lambda}}$ has at worst regular singularities. The polar divisor of $\nabla_{\boldsymbol{\lambda}}$ is equal to D_n;

(ii) The monodromy representation of $\nabla_{\boldsymbol{\lambda}}$ is conjugated to a representation

$$\pi_1(\mathbb{P}^1\setminus D_n)\longrightarrow \boldsymbol{D}_\infty.$$

The connection $\nabla_{\boldsymbol{\lambda}}$ is given by

$$\nabla_{\boldsymbol{\lambda}} = d+\begin{pmatrix}\mathcal{A}_{11} & \mathcal{A}_{12} \\ -\mathcal{A}_{21} & -\mathcal{A}_{11}\end{pmatrix}+\sum_{i=1}^{n-2}\begin{pmatrix}\mathcal{A}_{11}^i & \mathcal{A}_{12}^i \\ -\mathcal{A}_{21}^i & -\mathcal{A}_{11}^i\end{pmatrix},$$

where

$$\begin{aligned} \mathcal{A}_{11} &:= (x-1)\alpha_2(x,y)+\alpha_1(x,y)+\frac{1}{2}\frac{dy}{y}, \\ \mathcal{A}_{11}^i &:= (x-1)\alpha_2^i(x,y,z_i), \\ \mathcal{A}_{12} &:= \frac{dx+(x-1)^2\alpha_2(x,y)+2(x-1)\alpha_1(x,y)+\alpha_0(x,y)}{y}, \\ \mathcal{A}_{12}^i &:= \frac{(x-1)^2\alpha_2^i(x,y,z_i)+\alpha_0^i(x,y,z_i)}{y}, \\ \mathcal{A}_{21} &:= y\alpha_2(x,y), \\ \mathcal{A}_{21}^i &:= y\alpha_2^i(x,y,z_i), \end{aligned}$$

in the affine coordinates $[x:y:z_1:\dots:z_{n-2}:1]$.

Algebraic isomonodromic family

- Hyperplanes on $\mathbb{P}^n$:

$$\begin{cases} H_0 = (y-ax-bt=0) \\ H_i = (z_i-c_ix-d_it=0) \quad (i=1,2,\dots,n-2). \end{cases}$$

Here a,b,c_i, and d_i ($i=1,2,\dots,n-2$) are generic complex numbers. Let V be a Zariski open subset of $\operatorname{Spec}\mathbb{C}[a,b,c_i,d_i]_{i=1,\dots,n-2}$. Set

$$t_{2i+1}(V)+t_{2i+2}(V)=\mathcal{Q}_i|_{\cap_{i=0}^{n-2}H_i}, \qquad t_{2n-1}(V)=(x=0)|_{\cap_{i=0}^{n-2}H_i},$$
$$t_{2n+1}(V)=(y=0)|_{\cap_{i=0}^{n-2}H_i},\ t_{2n+1}(V)=(t=0)|_{\cap_{i=0}^{n-2}H_i}.$$

- Restrict the flat connection $\nabla_{\boldsymbol{\lambda}}$ to a generic line $\mathbb{P}^n\cap(\cap_{i=0}^{n-2}H_i)$. Then we have a connection:

$$(\nabla_{\mathbb{P}^1\times V/V})_{\boldsymbol{\lambda}}\colon F_0\longrightarrow F_0\otimes\Omega^1_{\mathbb{P}^1\times V/V}(D(\boldsymbol{t}))$$

where F_0 is the trivial rank 2 vector bundle over $\mathbb{P}^1\times V$.

Algebraic isomonodromic family

$(\nabla_{\mathbb{P}^1\times V/V})_{\boldsymbol{\lambda}}$ is an algebraic isomonodromic family of the Fuchsian systems with $(2\mathrm{n}-2)+3$ regular singularities:

$$d+\sum_{i=1}^{2n}A_i(V)\frac{dx}{x-t_i(V)}.$$

Reside matrices	$A_1(V)$	$A_2(V)$	$A_{2i+1}(V)$	$A_{2i+2}(V)$
Eigenvalues	$\pm\frac{1}{4}$	$\pm\frac{1}{4}$	$\pm\frac{\lambda_{i+1}}{2}$	$\pm\frac{\lambda_{i+1}}{2}$

$A_{2n-1}(V)$	$A_{2n}(V)$	$A_{2n+1}(V)$
$\pm\frac{\lambda_1}{2}$	$\pm\frac{\lambda_0-1}{2}$	$\pm\frac{\lambda_0+\lambda_1}{2}$

Since $\dim V = 2n-2$, this algebraic isomonodromic family gives an algebraic solution of some $(2n-2)$-variables Garnier system.

- By the Zariski's hyperplane section theorem and the Zariski–Van-Kampen method, we can compute a monodromy representation of the Fuchsian systems for some $\boldsymbol{v}\in V$:

γ_{t_1}	γ_{t_2}	$\gamma_{t_{2i+1}}$ ($i=1,\dots,n-2$)	$\gamma_{t_{2i+2}}$ ($i=1,\dots,n-2$)
$\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & a_0^2 \\ -a_0^{-2} & 0\end{pmatrix}$	$\begin{pmatrix}a_{i+1} & 0 \\ 0 & a_{i+1}^{-1}\end{pmatrix}$	$\begin{pmatrix}a_{i+1}^{-1} & 0 \\ 0 & a_{i+1}\end{pmatrix}$

$\gamma_{t_{2n-1}}$	$\gamma_{t_{2n}}$	$\gamma_{t_{2n+1}}$
$\begin{pmatrix}a_1 & 0 \\ 0 & a_1^{-1}\end{pmatrix}$	$\begin{pmatrix}-a_0 & 0 \\ 0 & -a_0^{-1}\end{pmatrix}$	$\begin{pmatrix}a_0a_1^{-1} & 0 \\ 0 & a_0^{-1}a_1\end{pmatrix}$

Here $a_j=\exp(-\pi\sqrt{-1}\lambda_j)$ $j=0,1,\dots,n-1$.

Construction

- **Step 1**.

$$\omega_0 := \lambda_0\left(\frac{du_0}{u_0}-\frac{du_1}{u_1}\right)+\lambda_1\left(\frac{du_0}{u_0-1}-\frac{du_1}{u_1-1}\right)$$

$$\psi_n := \begin{cases}\sum_{i=1}^{n-2}\lambda_{i+1}\left(\frac{d(u_0-u_1+z_i)}{u_0-u_1+z_i}-\frac{d(u_0-u_1-z_i)}{u_0-u_1-z_i}\right) & n>2 \\ 0 & n=2.\end{cases}$$

We have a family of flat connections

$$(\nabla_0)_{\boldsymbol{\lambda}} := d+\frac{1}{2}\begin{pmatrix}\omega_0+\psi_n & 0 \\ 0 & -\omega_0-\psi_n\end{pmatrix}.$$

- **Step 2**.

$$M_1(u_0,u_1) := \begin{pmatrix}-1 & -u_0+u_1 \\ -1 & u_0-u_1\end{pmatrix}.$$

$$\begin{aligned} \nabla_0' &:= d-\frac{1}{2}\frac{d(u_0-u_1)}{u_0-u_1}. \\ \nabla_0'' &:= d+M_1(u_0,u_1)^{-1}dM_1(u_0,u_1) \\ &\quad + M_1(u_0,u_1)^{-1}\frac{1}{2}\begin{pmatrix}\omega_0+\psi_n & 0 \\ 0 & -\omega_0-\psi_n\end{pmatrix}M_1(u_0,u_1). \end{aligned}$$

- **Step 3**. $\nabla_0''\otimes\nabla_0'$ decents to

$$\begin{aligned}(\nabla_1)_{\boldsymbol{\lambda}} = & d+\begin{pmatrix}\alpha_1(s_1,s_2) & \alpha_0(s_1,s_2) \\ -\alpha_2(s_1,s_2) & -\alpha_1(s_1,s_2)\end{pmatrix} \\ & +\sum_{i=1}^{n-1}\begin{pmatrix}0 & \alpha_0^i(s_1,s_2,z_i) \\ -\alpha_2^i(s_1,s_2,z_i) & 0\end{pmatrix}.\end{aligned}$$

under the map

$$(u_0,u_1,z_1,\dots,z_{n-2})\longmapsto[s_1:s_2:z_1:\dots:z_{n-2}:1],$$

where $s_1=u_0+u_1$ and $s_2=u_0u_1$. Here

$$\begin{aligned} \alpha_0(s_1,s_2) &:= \frac{2\lambda_0(1-s_1+s_2)+\lambda_1(-s_1+2s_2)}{2(1-s_1+s_2)}ds_1 \\ &\quad -\frac{\lambda_0s_1(1-s_1+s_2)+\lambda_1s_2(s_1-2)}{2s_2(1-s_1+s_2)}ds_2, \\ \alpha_0^i(s_1,s_2,z_i) &:= \lambda_{i+1}\left(dz_i-\frac{z_id(s_1^2-4s_2-z_i^2)}{2(s_1^2-4s_2-z_i^2)}\right), \\ \alpha_1(s_1,s_2) &:= -\frac{1}{4}\frac{d(s_1^2-4s_2)}{s_1^2-4s_2}, \\ \alpha_2(s_1,s_2) &:= -\frac{\alpha_0(s_1,s_2)}{s_1^2-4s_2}, \\ \alpha_2^i(s_1,s_2,z_i) &:= -\frac{\alpha_0^i(s_1,s_2,z_i)}{s_1^2-4s_2}. \end{aligned}$$

- **Step 4**. Put $x:=t-s_1+s_2$ and $y:=s_2$.

$$M_2(x,y) := \begin{pmatrix}y & x-1 \\ 0 & 1\end{pmatrix}.$$

$$\begin{aligned} \nabla_1' &:= d-\frac{1}{2}\frac{dy}{y} \\ \nabla_1'' &:= d+M_2(x,y)^{-1}dM_2(x,y) \\ &\quad + M_2(x,y)^{-1}\begin{pmatrix}\alpha_1(x,y) & \alpha_0(x,y) \\ -\alpha_2(x,y) & -\alpha_1(x,y)\end{pmatrix}M_2(x,y) \\ &\quad + \sum_{i=1}^{n-2}M_2(x,y)^{-1}\begin{pmatrix}0 & \alpha_0^i(x,y,z_i) \\ -\alpha_2^i(x,y,z_i) & 0\end{pmatrix}M_2(x,y). \end{aligned}$$

Desired flat connection

$$\nabla_{\boldsymbol{\lambda}} := \nabla_1''\otimes\nabla_1'.$$

References

[1] A. Girand, *A new two-parameter family of isomonodromic deformations over the five punctured sphere.* Bull. Soc. Math. France **144** (2016), no. 2, 339–368.

[2] A. Komyo, *A family of flat connections on the projective space having dihedral monodromy and algebraic Garnier solutions.* (arXiv:math/1806.00970)