# Moduli Space of Parabolic Higgs Bundles

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## History

- Metha and Seshadri introduced "parabolic structure" to vector bundles over Riemann surface, 1980.
- Generalized by Maruyama and Yokogawa to higher dimensional case, 1992.
- Simpson introduced the "Higgs field" for parabolic vector bundles over Riemann surface, 1990.
- Mochizuki studied higher dimensional parabolic Higgs bundles and showed the Kobayashi-Hitchin correspondence of parabolic version, 2006.
- Studied and generalized to other fields by many people, like Biswas, Borne, Donagi, Pantev, $\cdots$

## Parabolic Higgs Bundles

- Definitions given by Maruyama-Yokogawa (MY for simplicity): All divisors are combined together to define the parabolic structure given by filtration.
- Modified definitions are given by Mochizuki, Iyer-Simpson and Borne: Different filtrations for different components of the divisors to define the parabolic structure given by filtrations.
- $X$: connected smooth complex projective variety of dimension $n$.
- $D = \cup_{i=1}^k D_i \subset X$: normal crossing divisor with each component $D_i$ smooth and irreducible.

**Definition 1** (Mochizuki, Iyer-Simpson)**.** A parabolic sheaf on $(X, D)$ is a collection of torsion-free coherent sheaves $E_\alpha$ indexed by multi-indices $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Q}^k$ such that

- increasing: $E_\alpha \subset E_\beta$ whenever $\alpha \leq \beta$ (i.e. $\alpha_i \leq \beta_i$ for all $i$).
- normalization/support: $E_{\alpha+\delta^i} = E_\alpha(D_i) := E_\alpha \otimes \mathcal{O}_X(D_i)$, where $\delta^i = (0, \cdots, 1, \cdots, 0) \in \mathbb{Q}^k$ is a multi-index with the only 1 in the i-th position.
- semi-continuity: for any given $\alpha$, there exists a constant $c > 0$, such that for any multi-index $\varepsilon$ with each $0 \leq \varepsilon_i < c$, we have $E_{\alpha+\varepsilon} = E_\alpha$.

- The parabolic structure is completely determined by these $E_\alpha$ with each $a \leq \alpha_i < a+1$ for any $a \in \mathbb{Q}$.
- More precisely, it is completely determined by these $E_\alpha$ with $\alpha$ the jumping indices.
- The jumping indices are finite since $D$ has finite components.
- Parabolic weights: jumping indices $\alpha$ with each $0 \leq \alpha_i < 1$.
- Our definition of parabolic structure $\Leftrightarrow$ **Parabolic structure given by filtrations** (the first way to define the parabolic structure historically).
  For each divisor component, there is a filtration given as

$$0 = E_1^i \subset \cdots \subset E_{l_i}^i \subset E_{l_i+1}^i = E|_{D_i}$$

  and the associated graded term as ${}^iGr_j := E_{j+1}^i / E_j^i$ for $1 \leq j \leq l_i$, the corresponding parabolic weights are the rational numbers $j/l_i$ for these $j$ such that ${}^iGr_j \neq 0$.
- In MY case, parabolic structure is given by collection sheaves $\{E^a\}$ indexed by only one parameter $a \in \mathbb{Q}$ with $E^{a+1} = E^a(D)$.
- For the case of curves, modified definition completely equivalent to MY's.
- Can get a MY's structure by setting

$$E^a := E_\alpha, \quad \alpha = (a, \cdots, a).$$

- **Parabolic vector (or line) bundle** := parabolic sheaf with each component a vector (or line) bundle.
- Locally abelian parabolic bundle:= parabolic sheaf such that in a Zariski neighborhood of any point, it is isomorphic to a direct sum of parabolic line bundles.

- $(E, E_\alpha)$ locally abelian $\Rightarrow$ all $E_\alpha$ locally free.
- Locally abelian $\Rightarrow$ locally free, in the work of Iyer and Simpson.

**Example 2.** Let $\beta = (\beta_1, \cdots, \beta_k) \in \mathbb{R}^k$ be any multi-index, then we can define a natural parabolic line bundle denoted as $L := \mathcal{O}_X(\sum_{i=1}^k \beta_i D_i)$ by setting

$$L_\alpha := \mathcal{O}_X(\sum_{i=1}^k [\alpha_i + \beta_i] \cdot D_i),$$

where $[\cdot]$ denote the integral part.

**Definition 3.** A parabolic Higgs sheaf is given by the triple $(E, E_\alpha, \Phi)$ that consists of a parabolic sheaf $(E, E_\alpha)$ and a homomorphism $\Phi : E \to E \otimes \Omega_X^1(\log D)$ that satisfies:

- the natural composition $\Phi \wedge \Phi : E \to E \otimes \Omega_X^2(\log D)$ vanishes,
- $\Phi$ is a parabolic homomorphism, that is, $\Phi(E_\alpha) \subset E_\alpha \otimes \Omega_X^1(\log D)$ for all $\alpha$.

- Locally abelian parabolic Higgs bundle:= parabolic Higgs sheaf with the underlying parabolic sheaf locally abelian.
- Later on, always assuming locally abelian.

## Correspondences

- Simpson introduced the notion of parabolic Higgs bundle over non-compact curve (punctured Riemann surface) [Sim90] and obtained the non-abelian Hodge theory of non-compact version:

**Theorem 4** (Simpson)**.** Let $X$ be a compact Riemann surface with a divisor $D \subset X$ of finite points. Then there is a natural one-to-one correspondence between the category of stable parabolic Higgs bundles of degree 0 and the category of stable parabolic local systems of degree 0, or equivalently the category of tame harmonic bundles.

- Later studied and generalized by Mochizuki [Moc06, Moc09] to higher dimensional case and obtained the Kobayashi-Hitchin correspondence:

**Theorem 5** (Mochizuki)**.** Let $X$ be a smooth complex projective variety and $D \subset X$ a simple normal crossing divisor. A parabolic Higgs bundle over $(X, D)$ is polystable with degree 0 if and only if there exists a pluri-harmonic metric which is adapted to the parabolic structure, if and only if the parabolic flat bundle is polystable of degree 0.

- A vast development since 1990s.
- An important breakthrough by Biswas [Bis97] for the connection between parabolic vector bundles and orbifold vector bundles (vector bundles that equipped with a lift of the action of a Galois group):

$Y$ projective variety, $\Gamma$ Galois group (finite subgroup of Aut$(Y)$), $Y/\Gamma =: X$ assumed to be also smooth.

**Theorem 6** (Biswas)**.** There is an equivalence between the category of parabolic vector bundles on $X$ and orbifold vector bundles on $Y$.

- Biswas's idea refined by Borne (use MY's definition) [Bor07], Iyer-Simpson (use the refined definition) [Iy-Sim08]

**Theorem 7** (Borne,Iyer-Simpson)**.** There is an equivalence between the category of parabolic bundles on and the category of vector bundle on the associated root stack .

- $(X, D)$ as before, $\underline{n} = (n_1, \cdots, n_k)$ $k$-tuple of positive integers, called "denominators".
- $s_{D_i}$: tautological section of $\mathcal{O}_X(D_i)$ that vanishes along $D_i$ for each $i = 1, \cdots, k$.
- $L_{\mathbb{D}} := (\mathcal{O}_X(D_1), \cdots, \mathcal{O}_X(D_k))$, $\underline{s} := (s_{D_1}, \cdots, s_{D_k})$.
- Get root stack denoted $X[\frac{D_1}{n_1}, \cdots, \frac{D_k}{n_k}]$, consists of the pairs $(L_i, s_i)_{i=1}^k$, where each $L_i$ is a line bundle with $s_i \in H^0(X, L_i)$ such that $L_i^{\otimes n_i} \cong \mathcal{O}_X(D_i)$ and each $s_i^{\otimes n_i}$ corresponds to the canonical section $s_{D_i}$.
- Cadman: $Z := X[\frac{D_1}{n_1}, \cdots, \frac{D_k}{n_k}]$ is a Deligne-Mumford stack and the natural projection $p : Z \to X$ is finite and $X$ is the coarse moduli space for $Z$ under $p$.

## My Research

- Yokogawa [Yok93] constructed the moduli space of (semi-)stable parabolic Higgs bundles with fixed rational weights and Hilbert polynomial by introducing the parabolic Hilbert polynomial and the notion of (semi-)stability for parabolic Higgs bundles:

**Theorem 8** (Yokogawa)**.** There exists a quasi-projective scheme that parametrizes the (S-equivalence classes of) semi-stable parabolic Higgs bundles of fixed weights and Hilbert polynomial, and there is an open sub-scheme that parametrizes the stable ones. Moreover, the Hitchin morphism of parabolic Higgs bundles is projective.

- Yokogawa's results based on the classical geometric invariant theory, but parabolic structure considered by him was given by only one filtration and one sequence of weights.
- One direction of my work is to construct the moduli space of parabolic Higgs bundles in the modified sense and study its geometry.

Why the correspondence between parabolic bundles and vector bundles on the root stack important?

- Can study and find some properties of parabolic (Higgs) bundles by this equivalence, for example, Iyer and Simpson defined the parabolic Chern characters.
- The Kawamata covering of the root stack is a smooth variety, vector bundles on root stack correspond to some Galois group equivariant vector bundles, correspond to the orbifold bundles in Biswas's sense.
- Maybe very useful for the construction of the moduli space and have some application, for example, Bogomolov-Gieseker inequality.
- If the denominators go to infinity (adapted to all possible rational parabolic weights), get the "infinite root stack", which is pro-algebraic stack, in fact, a log space, this is studied in log geometry by Vistoli, Borne, Talpo, $\cdots$
- We can generalize this correspondence to the parabolic Higgs case:

**Proposition 9** (-)**.** There is an equivalence between the category of parabolic Higgs bundles on $(X, D)$ whose parabolic weights have denominators $n_1, \cdots, n_k$ and the category of Higgs bundles on $Z := X[\frac{D_1}{n_1}, \cdots, \frac{D_k}{n_k}]$. This equivalence perserves tensor products, direct sums and duals. Moreover, this equivalence preserves stability.

- Another direction of my research on parabolic Higgs bundles is to study how does the moduli space changes with the variation of parabolic weights, we know that the moduli space holds the same within the same chambers (walls that contribute by small change of weights ), but we don't know how does it cross there walls, this is the master space of parabolic Higgs bundles that proposed by Simpson.

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