On the Cohomology of the Moduli space of Parabolic Connections

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Introduction

In this poster, we study the moduli space of logarithmic connections of rank 2 on $\mathbb{P}^1 \setminus \{t_1, \dots, t_5\}$ with fixed spectral data. We compute the cohomology of such moduli space, and this computation will be used to extend the results of geometric Langlands corespondence of [1] to the case where the parabolic connections have five simple poles on $\mathbb{P}^1$ ([3]).

Preliminaries

We introduce $\mathfrak{sl}_2$-connections.
Fix complex numbers $\nu_1, \dots, \nu_n \in \mathbb{C}$. Suppose that $\nu_1 \cdots \nu_n \neq 0$ and

$$\sum_{i=1}^{n} \epsilon_i \nu_i \notin \mathbb{Z} \tag{1}$$

for any $(\epsilon_i), \epsilon_i \in \{1, -1\}$.

Definition 2.1 A *$\boldsymbol{\nu}$-$\mathfrak{sl}_2$-parabolic connection on* $\mathbb{P}^1$ is a triplet (E, ∇, φ) such that

1. E is a rank 2 vector bundle on $\mathbb{P}^1$,
2. $\nabla\colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ is a connection, where $D := t_1 + \cdots + t_n$,
3. $\varphi\colon \bigwedge^2 E \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ is a horizontal isomorphism,
4. the residue $\mathsf{res}_{t_i}(\nabla)$ of the connection ∇ at t_i has eigenvalues $\nu_i^{\pm}$, $1 \le i \le n$.

Here, we put

$$\nu_i^{\pm} := \pm \nu_i \quad (i = 1, \dots, n-1),\ \nu_n^{+} := \nu_n,\ \nu_n^{-} := 1 - \nu_n.$$

Denote by $\mathcal{M}_n$ the moduli stack of $\boldsymbol{\nu}$-$\mathfrak{sl}_2$-parabolic connections on $\mathbb{P}^1$, and by M_n its coarse moduli space. This moduli space is a smooth, irreducible quasi-projective algebraic variety of dimension $2(n-3)$.
For such $\boldsymbol{\nu}$, the parabolic direction $l_i := \ker(\mathbf{res}_{t_i}(\nabla) - \nu_i^{+}) \subset E_{|t_i}$ is uniquely determined. So, we can get the forgetful map

$$\mathsf{Bun} : M_n \to P_n \ ;\ (E, \nabla, \varphi) \mapsto (E, \{l_i\})$$

where P_n is the coarse moduli space of undecomposable quasi-parabolic bundles $(E, \{l_i\})$ on $\mathbb{P}^1$. Moreover, this morphism is a locally trivial affine $\mathbb{A}^{n-3}$-bundle.

Topologically, P_n is a non Hausdorff topological space, or a nonseparated scheme of dimension $n-3$.

Geometric Langlands Correspondence

Suppose $n = 4$. Then, P_4 is isomorphic to the projective line with doubled points $t_1, \dots, t_4$. Let $p : P_4 \to \mathbb{P}^1$ be the projection and denote by $t_i^{\pm} \in P_4$ the preimages of $t_i \in \mathbb{P}^1$. Let $[\nu] := \sum_{i=1}^{4} (\nu_i^{+} t_i^{+} + \nu_i^{-} t_i^{-}) \in \mathrm{div} P_4 \otimes_{\mathbb{Z}} \mathbb{C}$. Denote by D_ν the TDO ring corresponding to $[\nu]$.
Let $\xi = (E, \nabla)$ and $\xi_\nu := j_{!*}(\xi|_{\mathbb{P}^1 \setminus \{t_1, \dots, t_4\}})$. ξ_ν is a D_ν−module on P_4, where $j : \mathbb{P}^1 \setminus \{t_1, \dots, t_4\} \hookrightarrow \mathbb{P}^1$ is the natural injection.

Theorem 3.1 (D. Arinkin, [1]) Let $p_1 : \mathcal{M}_4 \times P_4 \to \mathcal{M}_4$ and $p_2 : \mathcal{M}_4 \times P_4 \to P_4$. Then, the functor

$$\Phi_{\mathcal{M}_4 \to P_4} : \mathcal{F} \mapsto Rp_{2,*}(\xi_\nu \otimes_{\mathcal{M}_4 \times P_4} p_1^* \mathcal{F})[1]$$

is an equivalence between $\mathcal{D}_{qc}(\mathcal{M}_4)^-$ and the derived category of D_ν−modules. The inverse functor is given by

$$\Phi_{P_4 \to \mathcal{M}_4} : \mathcal{F} \mapsto Rp_{1,*} \mathbb{D}\mathbb{R}_{P_4}((id_{\mathcal{M}_4} \times \sigma)^* \xi_\nu \otimes_{\mathcal{M}_4 \times P_4} p_2^* \mathcal{F})[1]$$

,where $\sigma : P_4 \to P_4$ is the involution such that $\sigma(t_i^{\pm}) = t_i^{\mp}$.

Conjecture 3.2 For arbitrary $n \ge 4$, there exists a categorical equivalence

$$\mathcal{D}_{qc}(\mathcal{M}_n)^- \xrightarrow{\sim} \mathcal{D}(P_n, D_\nu)$$

This conjecture can be considered as the special kind of Geometric Langlands Correspondence. Theorem 3.1 consider this correspondence as the Fourier - Mukai (- Laumon) transform.

Toward the $n = 5$ case : Poincaré bundle

In this section, we construct a Poincaré bundle (Fourier - Mukai kernel) for $n = 5$ case.
Suppose $n = 5$. Let $\hat{X}$ be the degree 4 del Pezzo surface, i. e., 5 points blow-up of $\mathbb{P}^2$. Then we have $P_5 = \hat{X} \cup X_0 \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$, where $X_i \simeq \mathbb{P}^2$.
There exists a double covering map $\Psi : \hat{X} \to \mathbb{P}^1 \times \mathbb{P}^1$, and associated involution $\tau : \hat{X} \to \hat{X}$. On $\mathbb{P}^1 \times \mathbb{P}^1$, there exist 10 special lines:

$$V_{t_i} = \{(z, w) \in \mathbb{P}^1_z \times \mathbb{P}^1_w \mid z = t_i\}, \quad H_{t_i} = \{(z, w) \in \mathbb{P}^1_z \times \mathbb{P}^1_w \mid w = t_i\}$$

The inverse images of $\{V_{t_i}, H_{t_i}\}$ with $t_i \in \{t_1, 0, 1, \infty\}$ correspond to 16 (-1)-curves $\{\Pi, \Pi_i, \Pi_{ij}\}$ on $\hat{X}$. Set $\Pi^{V_2, \pm} := \Psi^{-1}(V_{t_2})$, and $\Pi^{H_2, \pm} := \Psi^{-1}(H_{t_2})$. According to the parabolic structures, we associate 16 (-1)-curves $\{\Pi, \Pi_i, \Pi_{ij}\}$ with $\nu_i^{\pm}(i \neq 2)$, and rename them $\{\Pi^{V_i, \pm}, \Pi^{H_i, \pm}\}$. Set $[\nu] := \sum_{i=1}^{5} \{\nu_i^{+}(\Pi^{V_i,+} + \Pi^{H_i,+}) + \nu_i^{-}(\Pi^{V_i,-} + \Pi^{H_i,-})\}$ which is a $\mathbb{C}$-divisor on $\hat{X}$. Denote by D_ν the TDO ring corresponding to $[\nu]$.
For $(E, \nabla, \varphi) \in \mathcal{M}_5$, set $\mathcal{E} := \Psi^*(E \boxtimes E)$ as a $D_{\hat{X}}$−module. We denote by $\mathcal{E}_\nu$ the D_ν-module which is defined by $\mathcal{E}_\nu := j_{!*}(\mathcal{E}|_U)$, where $U := \hat{X} \setminus \{\Pi, \Pi^{V_2, \pm}, \Pi^{H_2, \pm}, \Pi_i, \Pi_{i,j}\}_{i=1,\dots,5}$, $j : U \hookrightarrow \hat{X}$ is the natural embedding. For $\mathbf{x} \in \hat{X}$, we have the bundle $\xi_{\mathbf{x}}$ on $\mathcal{M}_5$ whose fiber at (E, ∇, φ) is $E_{x_1} \otimes E_{x_2}$, where $\Psi(\mathbf{x}) = (x_1, x_2) \in \mathbb{P}^1 \times \mathbb{P}^1$. We may apply the minimal extension construction to the universal family ξ on $\mathcal{M}_5 \times \hat{X}$, and we obtain an $\mathcal{M}_5$-family ξ_ν of D_ν-modules. We may construct such twisted D-modules on each projective chart of P_5. By gluing them, we get a D_ν-module $\mathcal{E}_\nu$ on P_5 and $\mathcal{M}_5$-family ξ_ν of D_ν-modules on $\mathcal{M}_5 \times P_5$.

Main Result

Theorem 5.1 *Let* $\mathbf{x}, \mathbf{y} \in \hat{X}$. *Then*

$$H^i(\mathcal{M}_5, \mathcal{O}_{\mathcal{M}_5}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

$$H^i(\mathcal{M}_5, (\xi_\nu)_{\mathbf{x}} \otimes (\xi_\nu)_{\mathbf{y}}) = 0 \textit{ for any } \mathbf{y} \notin \{\mathbf{x}, \tau(\mathbf{x})\}, i \ge 0$$

The sketch of proof is as follows: we have a stratum M_5^0 as a Zariski open dense of M_5 where $E \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. There exists an injection $\iota : M_5^0 \hookrightarrow \mathrm{Hilb}(\mathcal{K}_5')$. Here, $\mathcal{K}_5'$ satisfies

$$H^i(\mathcal{K}_5', \mathcal{O}_{\mathcal{K}_5'}) = \begin{cases} \mathbb{C}, & i = 0, \\ H^2_{\mathfrak{m}}(A) \neq 0, & i = 1, \\ 0, & i \ge 2, \end{cases}$$

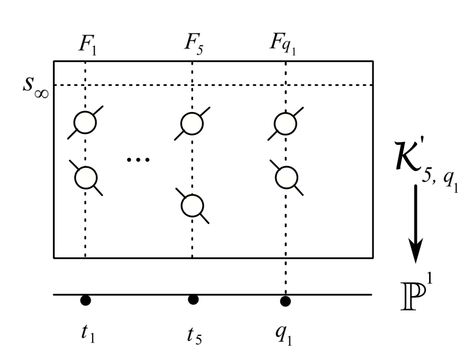
where $(A, \mathfrak{m})$ is a local ring such that $\dim A_{\mathfrak{m}} = 2$. We can define a map

$$f : \mathrm{Bl}_\Delta(\mathcal{K}_5' \times \mathcal{K}_5') \setminus T \longrightarrow \mathcal{K}_5'$$
$$(q_1, p_1, q_2, p_2) \longmapsto (q_1, p_1),$$

and the fiber $f^{-1}(\{(q_1, p_1)\}) \simeq \mathcal{K}_{5,q_1}'$. Then $\mathcal{K}_{5,q_1}'$ satisfies

$$H^i(\mathcal{K}_{5,q}', \mathcal{O}_{\mathcal{K}_{5,q}'}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

By using Leray's spectral sequence, we get our results.



[Figure 1.] The Surface $\mathcal{K}_5'$.

References

[1] D. Arinkin, *Orthogonality of natural sheaves on moduli stacks of SL(2)-bundles with connections on P1 minus 4 points.*, Selecta Math., New Series 7 (2001), 213-239.

[2] A. Komyo, M.-H. Saito, *Explicit description of jumping phenomena on moduli spaces of parabolic connections and Hilbert schemes of points on surfaces, accepted in Kyoto Journal of Mathematics.* (arXiv:math/1611.00971)

[3] Y. Matsubara. *An Example of The Langlands Correspondence for Regular Rank Two Connections on* $\mathbb{P}^1$. (in preparation)