Maurer-Cartan equation and scattering diagrams: the work of Chan, Leung and Ma and possible extensions

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Abstract

Recent work of Chan, Leung and Ma [CLM] analyses the asymptotic behaviour of certain special solutions to the Maurer-Cartan equation which governs first order deformations of a semi-flat Calabi-Yau manifold, through a Fourier transform. Their main result is that the leading order asymptotics defines naturally a consistent scattering diagram in the sense of Kontsevich-Soibelman and Gross-Siebert. Our goal is to prove an analogous result for solutions to the Maurer-Cartan equation which governs first order deformation of a complex manifold and a holomorphic vector bundle on it.

Setting

Let B be a tropical affine manifold, Λ be a lattice subbundle of TB and $\Lambda^* = Hom_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ the dual lattice. Then, set:

$\check{\boldsymbol{X}} := \boldsymbol{TB/\Lambda}$, $\check{p} : \check{X} \to B$

$J_\hbar := \begin{pmatrix} 0 & \hbar I \\ -\hbar^{-1} I & 0 \end{pmatrix}$

is a family of complex structures, written in the base $\partial_{x_1}, \dots, \partial_{x_n}, \partial_{\check{y}_1}, \dots, \partial_{\check{y}_n}$, where $\{\check{y}_j\}$ are coordinates of the fibers of $\check{p}$.

$\boldsymbol{X := T^*B/\Lambda^*}$, $p : X \to B$

$\omega_\hbar := \hbar^{-1} dy_j \wedge dx^j$

is a family of symplectic forms, where $\{y_j\}$ are coordinates of the fibers of p.

Mirror dgLa

Kodaira-Spencer dgLa

$(\Omega^{0,\bullet}(\check{X}, T^{1,0}_{\check{X}}), \bar{\partial}_{\check{X}}, [\cdot,\cdot])$

$\bar{\partial}_{\check{X}}$ is the complex structure on $\check{X}$ compatible with $J_\hbar$

$[\cdot,\cdot]$ is the Lie bracket on $T^{1,0}\check{X}$

$\nabla^{1,0}$ is the holomorphic connection on $T^{1,0}_{\check{X}}$

Fourier transform

Symplectic dgLa

$(\Omega^\bullet(\mathcal{M}, T\mathcal{M}), d_W, \{\cdot,\cdot\})$

$\mathcal{M} = \bigsqcup_{x \in B} \pi_1(p^{-1}(x), s(x))$,

s is the zero section

$d_W := d - 2\pi s \lrcorner \omega_\hbar$

$\{X, Y\} = \iota_X * \nabla^W(Y) - \iota_Y * \nabla^W(X)$

$\nabla^W := \nabla + 2\pi s \lrcorner \omega_\hbar$

∇ is the flat connection on B

$pr : \mathcal{M} \to B$ is locally trivial, i.e. for $U \subset B$ $pr^{-1}(U) \cong U \times \mathbb{Z}^n$ and we can choose coordinates (x, m), such that $m \in \mathbb{Z}^n$ represent an affine loop in $\check{p}^{-1}(U)$ with tangent vector $\sum_j m_j \frac{\partial}{\partial y_j}$. There is also a natural identification $\Omega^\bullet(U_m, T\mathcal{M}) \cong \Omega^\bullet(U, TB)$, where $U_m = U \times \{m\}$.

$$\mathcal{F}(\alpha(x,\check{y}))_m := \left(\frac{2\pi}{\hbar}\right)^{|\alpha|} \int_{\pi^{-1}(x,m)} \alpha_I^j(x,\check{y}) e^{-2\pi i (m,\check{y})} d\check{y} dx_I \otimes \frac{\partial}{\partial x_j}, \quad \pi : \mathcal{M} \times_B X \to \mathcal{M}$$

$$\mathcal{F}^{-1}(\alpha(x,m))(\check{y}) = \left(\frac{2\pi}{\hbar}\right)^{-|\alpha|} \sum_{m \in \mathbb{Z}^n} \alpha_I^j(x,m) e^{2\pi i (m,\check{y})} d\bar{z}_I \otimes \frac{\partial}{\partial z_j}$$

Scattering Diagrams

Let $\mathfrak{g} := (\mathbb{C}[\Lambda] \hat{\otimes}_{\mathbb{C}} \mathbb{C}[[t]])(t) \otimes_{\mathbb{Z}} \Lambda^*$, then $G := \exp(\mathfrak{g})$ is a subgroup of $Aut_{\mathbb{C}[[t]]}(\mathbb{C}[\Lambda] \hat{\otimes}_{\mathbb{C}} \mathbb{C}[[t]])$. The tropical vertex group ([GPS]) is defined as $\mathbf{V}{:=}\exp(\mathfrak{h}) \subset G$, where $\mathfrak{h} = \bigoplus_{m \neq 0} z^m (t \otimes m^\perp) \subseteq \mathfrak{g}$.

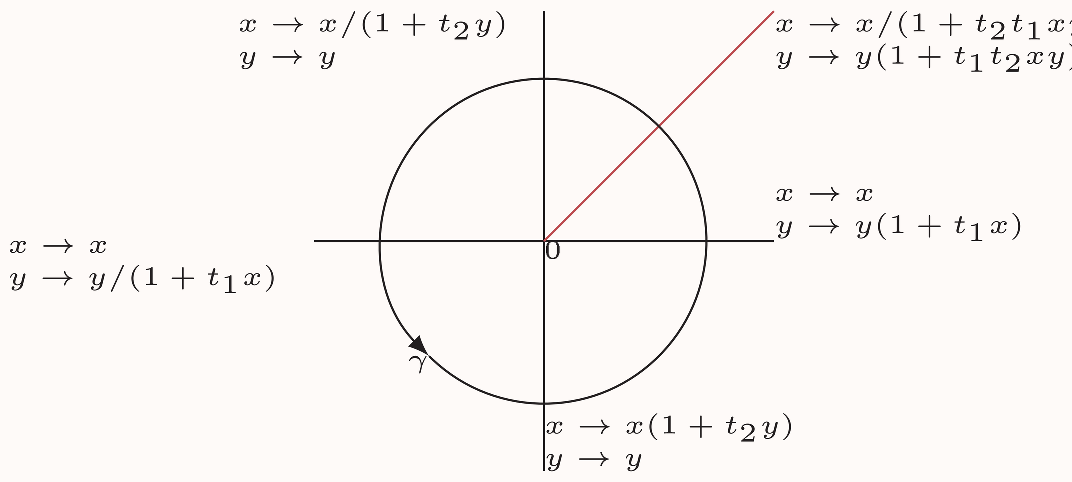
Scattering diagrams are a way to represent the tropical vertex group. They are a collection of *walls* $\mathbf{w}_i = (\sigma_i, f_i)$:

- σ_i is either a *ray* ($\sigma_i = m_0 + \mathbb{R}_{\geq 0} m_i$) or a *line* ($\sigma_i = m_0 + \mathbb{R} m_i$),
- $f_i \in \mathbb{C}[z^{m_i}] \hat{\otimes}_{\mathbb{C}} \mathbb{C}[[t]]$ and $f_i \equiv 1 \bmod z^{m_i}(t)$.

Given a loop γ there exists a notion of *path ordered product* $\Theta_{\gamma,\mathcal{D}}$. For instance if $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2\}$, then $\Theta_{\gamma,\mathcal{D}} := \theta_2^{-1} \circ \theta_1^{-1} \circ \theta_2 \circ \theta_1$, for $\theta_i = \exp(\log(f_i) \otimes n_i)$, $n_i \in \Lambda^*$ primitive and $\langle n_i, m_i \rangle = 0$.

Theorem ([KS]). *Let $\mathcal{D}$ be a scattering diagram with two walls. There exists a unique minimal scattering diagram $S(\mathcal{D}) \supset \mathcal{D}$ such that $S(\mathcal{D}) \setminus \mathcal{D}$ consists only of rays, and it is* consistent, *meaning* $\Theta_{\gamma, S(\mathcal{D})} =$ Id, *for any loop γ intersecting $S(\mathcal{D})$ generically.*

$\mathcal{D} = \{(\mathbb{R}(1,0), 1 + t_1 x), (\mathbb{R}(0,1), 1 + t_2 y)\}$



$S(\mathcal{D}) \setminus \mathcal{D} = \{(\mathbb{R}_{\geq 0}(1,1), 1 + t_1 t_2 xy)\}$

Maurer-Cartan equation and consistent scattering diagram

First order deformation of $\check{X}$ are determined by solutions of the Maurer-Cartan equation $\bar{\partial}\psi + [\psi, \psi] = 0$, for $\psi \in \Omega^{0,1}(\check{X}, T^{1,0}\check{X})[[\mathbf{t}]]$. Let $\mathfrak{g} = \Omega^0(\check{X}, T^{1,0}_{\check{X}})[[\mathbf{t}]](\mathbf{t})$, then $G := \exp(\mathfrak{g})$ acts on $\Omega^{0,\bullet}(\check{X}, T^{1,0}_{\check{X}})[[\mathbf{t}]]$ by $e^{\check{g}} * \psi := \sum_{k \geq 0} \frac{[\check{g}, \psi]^k}{k!}$. Since $\check{X}(U)$ has no non trivial deformations, every $\psi \in \Omega^{0,1}(\check{X}, T^{1,0}_{\check{X}})$ can be written locally as $e^{\check{g}} * 0$. We are interested in solutions $e^{\check{g}} * 0$, where $\check{g}$ has, locally, an assigned behaviour in the limit $\hbar \to 0$. In particular, we set $\lim_{\hbar \to 0} \check{g} := \begin{cases} Log(\theta) & on H_+ \\ 0 & on H_- \end{cases}$, where $\theta = \theta_{m, f_m}$ and H_+, H_- are the two half spaces divided by the *line* $\sigma = m_0 + \mathbb{R}m$, according to the orientation of σ.
Working on the symplectic side (via the Fourier transform defined above), they construct a solution Π to the Maurer-Cartan equation $d_W \Pi + \{\Pi, \Pi\} = 0$, such that it is supported on σ. Then, they look for $g \in \Omega^0(\mathcal{M}, T\mathcal{M})$, such that $e^g * 0 = \Pi$ and g has the desired asymptotic behaviour.
Let Π_1 and Π_2 be solutions to Maurer-Cartan equation supported on two distinct lines σ_1, σ_2, then $\Pi := \Pi_1 + \Pi_2$ is used as input to construct the solution Φ (even if Π is not a solution). It turns out that Φ can be decomposed as a sum of its Fourier modes Φ_m and each Φ_m solves Maurer-Cartan equation and it is supported on a ray σ_m. Studying the asymptotic behaviour of g_m such that $e^{g_m} * 0 = \Phi_m$ they associate to each Φ_m the wall $\mathbf{w}_m = (\sigma_m, \lim_{\hbar \to 0} g_m$ and this procedure gives rise to a consistent scattering diagram.

Developments

Right now, we understand the symplectic dgLa, mirror to the Kodaira-Spencer dgLa that governs deformations of holomorphic pair $(\check{X}, \check{L})$, in the case when $\check{L}$ is a line bundle over $\check{X}$.

$$(\Omega^{0,\bullet}(\check{X}, \mathcal{O}_{\check{X}} \oplus T^{1,0}_{\check{X}}), \bar{\partial} = \begin{pmatrix} \bar{\partial}_{\check{X}} & B \\ 0 & \bar{\partial}_{\check{X}} \end{pmatrix}, [\cdot,\cdot]) \quad \to \quad (\Omega^\bullet(\mathcal{M}, \mathcal{O}_{\mathcal{M}} \oplus T\mathcal{M}), d = \begin{pmatrix} d_W & \hat{B} \\ 0 & d_W \end{pmatrix}, \{\cdot,\cdot\})$$

$B\psi := -(-1)^{|\psi|} \psi \lrcorner F_{\check{L}}$, where $F_{\check{L}}$ is the Chern curvature of $\check{L}$, and the Lie bracket is $[(\xi, \psi), (\eta, \varphi)] := (\psi \lrcorner \partial\eta - (-1)^{|\xi||\eta|} \varphi \lrcorner \xi + \xi \wedge \eta, [\psi, \varphi]_{T^{1,0}\check{X}})$.

$\hat{B}X(m,x) := -(-1)^{|X|} \sum_n X_{ij}(n,x) \left(\int F_{jk} e^{-2\pi i (m-n, \check{y})} d\check{y} \right) dx_i \wedge dx_k$, and the Lie bracket is

$$\{(\alpha, X), (\beta, Y)\}_m := \left(\sum_n \alpha(n) \wedge (\hat{B}Y)_{m-n} - (-1)^{|\alpha||\beta|} \sum_n \beta(n) \wedge (\hat{B}X)_{m-n} + \alpha \wedge \beta, \{X, Y\}_m \right).$$

References

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