Toward a Frobenius Structure on a Stability Manifold

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Abstract

We use Saito's theory of semiuniversal unfolding of isolated singularities to put a F-structure on the space of stability conditions of a Picard-rank-20 K3 surface away from codimension one subset.

Background and Motivation

- Classical mirror symmetry predicts that a Calabi-Yau manifold with complex and symplectic structures (X, I, ω) admits a (family of) mirror(s) $(\hat{X}, \hat{I}, \hat{\omega})$, such that

$$GW(X, \omega) \Longleftrightarrow VHS(\hat{X}, \hat{I}).$$

  Such an equivalence can be crystallized as an isomorphism of Frobenius structures on (germs of) the so-called Kähler moduli and complex moduli, which parameterize mirror families of $(\hat{X}, \hat{I})$ and (X, ω), respectively.
- Now, Kontsevich's homological mirror symmetry suggests that we should consider more broadly derived categories (and Fukaya categories) instead of manifolds.
- Spaces of stability conditions of these categories are heuristically global versions of Kähler (and complex) moduli, parameterizing mirror categories.
- A "global" Frobenius structure is speculated to exist on a stability manifold [KKP08]. A fascinating approach was carried out by Bridgeland, Stoppa and their collaborators [BL12] [BSS16], based on wall-crossing formula for DT-invariants.

Goal of this Project

Our goal is to understand the above idea in a particular example, namely, to construct a Frobenius structure on the space of stability conditions $Stab(S_{20})$ of a particular Picard-rank-20 $K3$ surface S_{20}.

The $K3$ Surface S_{20}

S_{20} is the Picard-rank-20 $K3$ surface whose transcendental lattice $T(S_{20}) := NS(S_{20})^{\perp} \subset H^2(S_{20}, \mathbb{Z})$ is

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

As shown in [SS17], S_{20} is the mirror of those $K3$ categories $\mathcal{T}_Y$ associated to very general cubic fourfolds Y.

Kuznetsov's $K3$ Categories $\mathcal{T}_Y$

The derived category $D^b(Y)$ of a smooth cubic fourfold Y decomposes

$$D^b(Y) = \langle \mathcal{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

into a relatively simple part (an exceptional collection) and a highly nontrivial subcategory

$$\mathcal{T}_Y := \{E \in D^b(Y) : R^{\cdot}Hom(\mathcal{O}_Y(i), E) = 0,\ i = 0, 1, 2\},$$

called the Kuznetsov component.

$\mathcal{T}_Y$ is a K3 category/noncommutative K3 in the sense that:

- Its Serre functor is $(-)[2]$,
- Its Hochschild cohomology is the same as that of a $K3$ surface.
- There exists a family of these, parameterized by the moduli (stack) of cubic fourfolds $\mathcal{M}^{\circ}_{cubic4}$. In this family, there are countably many (Hassett) divisors corresponds to geometric Kuznetsov components, i.e. $\mathcal{T}_Y \cong D^b(K3)$, while for a very general Y, $\mathcal{T}_Y$ is not geometric and could be viewed as noncommutative deformation of $K3$ surfaces [Kuz10].

Bridgeland Stability Conditions

A Bridgeland stability condition on $D^b(S_{20})$ is a pair $(\mathcal{A}, Z)$, where

- $\mathcal{A}$ is a heart of a bounded t-structure of $D^b(S_{20})$, and
- $Z : N(S_{20}) := H^0 \oplus NS \oplus H^4 \to \mathbb{C}$ is a group homomorphism,

satisfying some (rather strong) axioms.

Examples

		$Z_{\beta,\omega}(E) := -\int_X exp(-\beta - i\omega)ch(E)$,
$\beta + i\omega \in NS(S_{20})_{\mathbb{C}}$		$\mathcal{A}_{\beta,\omega}$:= tilted category of $Coh(S_{20})$
with ω ample	$\longrightarrow$	w.r.t. Mumford slope $\dfrac{c_1(E).\omega}{r(E)} = \beta.\omega$.
(Kähler cone)		(Geometric stability conditions)

Theorem ([Bri07])**.** *The set of all stability conditions $Stab(S_{20})$ of $D^b(S_{20})$ admits a complex manifold structure.*

A Map from $Stab(S_{20})$ to $\mathcal{M}_{cubic4}$

- $\mathcal{M}_{cubic4}$ is the moduli space of projective cubic fourfolds with at worst ADE singularities.
- Observe that we have an abstract isometry

$$N(S_{20}) := H^0 \oplus NS \oplus H^4 \cong H^4_{prim}(Y, \mathbb{Z}).$$

- With this lattice coincidence, and using [Bri08] and [Laz07], we have a map:

$$\begin{array}{ccc} Stab(S_{20}) & \ni & (\mathcal{A}, Z) \\ \downarrow & & \downarrow \\ P \subset N(S_{20})_{\mathbb{C}} & \ni & Z \\ \downarrow /GL_2^+ & & \\ D & & \\ \downarrow /\Gamma & & \\ \mathcal{M}_{cubic4} & & \end{array}$$

$D := \{\Omega \in \mathbb{P}(H^4_{prim}(Y, \mathbb{C})) : (\Omega, \Omega) = 0, (\Omega, \bar{\Omega}) > 0\} \setminus$ Hyperplanes

, where $\Gamma \subset Aut(H^4_{prim}(Y, \mathbb{Z}))$

The first map is a local homeomorphism, and a covering over its image (Bridgeland).

P is a $GL_2^+(\mathbb{R})$-bundle over D, the period domain.

D/Γ is the moduli of cubic fourfolds: global Torelli theorem for cubic fourfolds (Laza).

What we understand so far

Proposition 1. *Let $\mathcal{M}_{sm} \subset \mathcal{M}_{cubic4}$ be the locus of smooth cubic fourfolds, define $U := \pi^{-1}(\mathcal{M}_{sm}) \subset Stab^{\dagger}(S_{20})$. Then there exists a F-structure $(\circ, e, E)$ on U.*

Main Ingredient: Saito's Frobenius Structures

Recall that given a polynomial f which defines an isolated singularity, the (base space of) semiuniversal unfolding M of f carries a Frobenius structure $(\circ, e, E, g)$ [Sai81; Sai83] [Sai89].

Essentially, the product $\circ$ on $U \subset Stab(S_{20})$ coming from that of a subring $J_{mod3}(f)$ of the Jacobian ring $J(f)$ of a homogeneous polynomial f defining a smooth cubic fourfold Y_f :

$$J_{mod3}(f) := J_0 \oplus J_3 \oplus J_6 \subset J(f) := \mathbb{C}[x_0, \ldots x_5]/(\partial_i f) = \bigoplus_{i=0}^{6} J_i,$$

via various identifications

$$J_{mod3}(f) \cong H^4_{prim}(Y_f, \mathbb{C}) \cong N(S_{20})_{\mathbb{C}} \cong T_{\sigma_f}Stab(S_{20}),$$

where $\sigma_f \in U$ is a preimage of $Y_f \in \mathcal{M}_{sm}$ under π.

Further Questions

- Can we construct a flat metric g on U that together with $(\circ, e, E)$ form a Frobenius structure?
- Can we extended any of the structures across the divisor D over the locus of singular cubic fourfolds?
- Looking forward, would a global Frobenius structure on $Stab(S_{20})$, if exists, give away any information about the group of derived autoequivalences of S_{20} or even other $K3$ surfaces?

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