# Homology of moduli spaces of complexes

Jacob Gross
Oxford University
(PhD student)

## Motivation

It is already known, from the works of Kontsevich–Soibelman and Joyce–Song, that the wall-crossing formula for Donaldson–Thomas invariants of Calabi–Yau 3-folds can be expressed solely in terms of the Lie bracket of a Ringel–Hall algebra of stack functions. It is conjectured that all enumerative invariants with wall-crossing (Mochizuki invariants of algebraic surfaces, Seiberg–Witten invariants of 4-manifolds with $b_+ = 1$, Donaldson–Thomas invariants of Fano 3-folds, Donaldson–Thomas of Calabi–Yau 4-folds, and putative Donaldson–Segal invariants of $G_2$-manifolds) change according to this same universal formula, with an appropriate Lie bracket. Joyce proved that the homologies of moduli stacks of objects in certain dg-categories are graded vertex algebras. Using a procedure of Borcherds, one can extract a natural graded Lie algebra from a graded vertex algebra. For example, the graded Lie algebra defined on the homology of the moduli stack of objects in the derived category of representations of the $A_1$ quiver is $\mathfrak{sl}(2,\mathbb{C})$. One may hope that the wall-crossing formulae are written in terms of this graded Lie bracket.

## Future Directions

- Borcherds and Li have notions of a vertex algebra twisted by a formal group law $F$. We conjectured that, for a complex oriented cohomology theory $E$, the $E$-homology theory of a vertex type moduli stack is an $F$-vertex algebra. This may help us prove wall-crossing formulas for invariants in K-theory, elliptic cohomology, or complex cobordism. A wall-crossing forumla for complex cobordism Mochizuki invariants may have applications to the Göttsche–Kool conjecture on algebraic cobordism invariants of algebraic surfaces.
- Compute the homology of the moduli stack of dimension zero sheaves $\mathcal{M}^{\dim 0}$ on an algebraic surface $S$. We then hope to establish a Kirwan surjectivity result, that there is an injection
$$H_*(\textstyle\bigoplus_{n\geq 0}\mathrm{Hilb}^n(S)) \hookrightarrow H_*(\mathcal{M}^{\dim 0}).$$
This may help to explain the Grojnowski–Nakajima results on the Heisenberg algebra action on $H_*(\oplus_{n\geq 0}\mathrm{Hilb}^n(S))$.

## Why do vertex algebras appear in moduli theory?

In a recent incomplete preprint [2] Joyce proved that the homologies of moduli stacks $\mathcal{M}$ of objects in certain dg-categories $\mathcal{A}$ are graded vertex algebras. This is done by writing out an explicit state-to-field correspondence which depends on the existence of the following extra data (which are subject to some conditions):

- A quotient $K(\mathcal{A})$ of the Grothendieck group $K_0(\mathcal{A})$
- signs $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ for all $\alpha, \beta \in K(\mathcal{A})$
- a symmetric $\mathbb{Z}$-bilinear form $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$
- a complex $\Theta^\bullet \in \mathrm{Perf}(\mathcal{M} \times \mathcal{M})$

The natural choice of $\Theta^\bullet$ is simplest when $\mathcal{A}$ is $2n$-Calabi-Yau–so that $\mathcal{M}$ is $(2-2n)$-shifted symplectic. But a natural choice of $\Theta^\bullet$ always exists because $T^*\mathcal{M}[2m]$ is always even-shifted symplectic and $H_*(T^*\mathcal{M}[2m]) \cong H_*(\mathcal{M})$.
Borcherds proved that a vertex algebra can be constructed on any bialgebra that is equipped with a derivation and a compatible bicharacter.

- **Bialgebra:** The topological realization of the moduli stack of objects in any triangulated or abelian dg-category is an H-space under direct sum of objects. The homology of an H-space is a graded bialgebra.
- **Derivation:** Linearity of $\mathcal{A}$ induces a stack morphism $[*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$. Taking homology gives a map $R[t] \to \mathrm{End}(H_*(\mathcal{M}))$. The action of $t$ is an even derivation.
- **Bicharacter:** We expect that
$$r(v,w) = t^{\chi(\alpha,\beta)} \cdot (v \boxtimes w \frown \exp(\mathrm{ch}(\Theta^\bullet_{\alpha,\beta})))$$
is a compatible bicharacter.

### Main Theorem

Let $X$ be a smooth complex projective $\mathbb{C}$-variety and let $\underline{\mathrm{Perf}}(X)$ denote the derived $\mathbb{C}$-stack of perfect complexes on $X$. Then there is an isomorphism of graded $\mathbb{Q}$-Hopf-algebras
$$H_*(\underline{\mathrm{Perf}}(X),\mathbb{Q}) \cong \mathbb{Q}[K_0^{\mathrm{sst}}(X)] \otimes \mathbb{Q}[\textstyle\bigoplus_{q\geq 1, n\geq 0} L^q H^{2q-n}(X,\mathbb{Q})],$$
where $L^*H^*(X,\mathbb{Q})$ denotes the rational morphic cohomology of $X$ and $K_0^{\mathrm{sst}}(X)$ denotes the 0th semi-topological K-group of $X$.

Moreover, if $X$ is a smooth complex projective curve, surface, or toric 3-fold then $H_*(\underline{\mathrm{Perf}}(X),\mathbb{Q})$ is isomorphic, as a graded $\mathbb{Q}$-Hopf-algebra, to
$$\mathbb{Q}[K^0_{\mathrm{top}}(X^{\mathrm{an}})] \otimes \mathrm{Sym}(H^{\mathrm{even}}(X^{\mathrm{an}}) \otimes t^{-1}\mathbb{Q}[t^{-1}]) \otimes \Lambda(H^{\mathrm{odd}}(X^{\mathrm{an}}) \otimes t^{-\frac{1}{2}}\mathbb{Q}[t^{-\frac{1}{2}}]).$$
Algebras of this form can be given the structure of a lattice vertex super algebra. The presenter has not yet shown that, in this case, Joyce's operations agree with the usual vertex algebra operations but he hopes to prove this soon.

## Techniques used

- Blanc's theorem that
$$\mathcal{M}^{\mathrm{an}} \simeq \Omega^\infty \mathbf{K}^{\mathrm{sst}}(\mathrm{Perf}(X)).$$
- A theorem of Antieu–Heller that
$$\mathbf{K}^{\mathrm{sst}}(\mathrm{Perf}(X)) \simeq \Omega^\infty K^{\mathrm{sst}}(X).$$
- For all connected components $\mathcal{M}_\alpha$ of $\mathcal{M}$, there is an $\mathbb{A}^1$-homotopy equivalence $\mathcal{M}_\alpha \simeq \mathcal{M}_0$.
- Using Sullivan minimal model theory, one can show that $H_*(\Omega^\infty K^{\mathrm{sst}}(X)_0,\mathbb{Q})$ is the free commutative-graded algebra on the rational homotopy groups of $\Omega^\infty K^{\mathrm{sst}}(X)_0$ which are, by definition, the groups $K_n^{\mathrm{sst}}(X)_{\mathbb{Q}}$.

## Kirwan Surjectivity

Heinloth recently computed the homology of the moduli space $\underline{\mathrm{Coh}}(C)$ of coherent sheaves on a smooth projective complex curve $C$. Using this, and this poster's main theorem, we are able to establish that there is an injection
$$H_*(\underline{\mathrm{Coh}}(C),\mathbb{Q}) \hookrightarrow H_*(\underline{\mathrm{Perf}}(C),\mathbb{Q})$$
in rational homology.
Now, $H_*(\underline{\mathrm{Perf}}(C),\mathbb{Q})$ is a lattice vertex algebra and lattice vertex algebras are known to come from a Borcherds bicharacter construction. Therefore, we can write down an explicit vertex algebra structure on $H_*(\underline{\mathrm{Coh}}(C),\mathbb{Q})$ by restricting the bicharacter.

## Orientability of DT4 moduli spaces

In the process of proving the main theorem on this poster, we realized that Blanc's semi-topological K-theory of complex non-commutative spaces could also be used to solve the orientation problem for moduli spaces of sheaves on Calabi–Yau 4-folds (see [1]). The moduli space of perfect complexes of coherent sheaves $\mathcal{M}$ on $X$ is equivalent to the semi-topological K-theory space $\Omega^\infty K^{\mathrm{sst}}(X)$ and the moduli space of stabilized unitary connections $\mathcal{B}$ is equivalent to $\Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$. The natural orientation bundle on $\mathcal{B}$ (coming from the Dirac operator) pulls back to the natural orientation bundle on $\mathcal{M}$ (coming from the $-2$-shifted symplectic structure) along the K-theory comparison map $\Omega^\infty K^{\mathrm{sst}}(X) \to \Omega^\infty K^{\mathrm{top}}(X^{\mathrm{an}})$. Cao–Joyce proved orientability of moduli spaces of connections on spin 8-manifolds–this then gives orientations on $\mathcal{M}$ and the substack $\mathcal{M}^{\mathrm{coh}} \subset \mathcal{M}$ of coherent sheaves on $X$. This has applications to the problem of defining invariants that 'count' semi-stable coherent sheaves of a fixed topological type on Calabi–Yau 4-folds.

## References

[1] Y. Cao, J. Gross, and D. Joyce, *On orientations for moduli spaces of coherent sheaves on Calabi–Yau manifolds*, December 2018, preprint.

[2] D. Joyce, *Ringel–Hall style vertex algebra and Lie algebra structures on the homology of moduli spaces*, April 2018, preprint.

## Acknowledgements

The presenter extends his biggest thanks to his supervisor Dominic Joyce. He also thanks Chris Brav, Yalong Cao, André Henriques, Francis Kirwan, Martijn Kool, David Rydh, Jan Steinebrunner, Richard Thomas, and Bertrand Töen for useful discussions. The presenter acknowledges support from the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics during the time this work was completed.

## Contact Information

- Web: http://users.ox.ac.uk/∼ linc4221/
- Email: Jacob.Gross@maths.ox.ac.uk