Perversity equals weight for Painlevé spaces

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Introduction

Fix a smooth projective curve C over $\mathbb{C}$, integers $r \geq 2$ and $d \in \mathbb{Z}$, points $p_1, \dots p_n \in C$, for each p_j choose a local coordinate z_j and an irregular type $Q_j \in \mathfrak{t} \otimes z_j^{-1}\mathbb{C}[z_j^{-1}]$ (where $\mathfrak{t} \subset \mathfrak{gl}(r, \mathbb{C})$ is the standard Cartan subalgebra), a parabolic subalgebra $\mathfrak{t} \subset \mathfrak{p}_j \subset \mathfrak{gl}(r, \mathbb{C})$ and an adjoint orbit $\mathcal{O}_j$ in its Levi factor $\mathfrak{l}_j$. Moreover, we fix parabolic weights $0 \leq \alpha_j^1 < \cdots < \alpha_j^{l_j} < 1$ for the non-trivial summands of $\mathfrak{l}_j$. To this data, it is possible to associate the following spaces:

- a wild character variety $\mathcal{M}_{\mathrm{B}}$ of generalized monodromy representations of the fundamental group of C (complemented by Stokes matrices) [2];
- a de Rham moduli space $\mathcal{M}_{\mathrm{dR}}$ of α-semistable parabolic connections with irregular singularities at $p_1, \dots p_n$ [1];
- a Dolbeault moduli space $\mathcal{M}_{\mathrm{Dol}}$ of α-semistable parabolic meromorphic Higgs bundles with higher-order poles at $p_1, \dots p_n$ [1].

Among these spaces, we have the following maps:

- Irregular Riemann–Hilbert correspondence: a $\mathbb{C}$-analytic isomorphism $\mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{B}}$;
- Wild non-abelian Hodge theory: a diffeomorphism $\mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{Dol}}$;
- combining the previous two maps: a diffeomorphism $\Psi : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{M}_{\mathrm{B}}$.

On the spaces we have the following structure:

- on $\mathcal{M}_{\mathrm{B}}$: structure of an affine variety over $\mathbb{C}$;
- on $\mathcal{M}_{\mathrm{Dol}}$: algebraically completely integrable system, called irregular Hitchin map $h : \mathcal{M}_{\mathrm{Dol}} \to Y$ where Y is an affine space of $\dim Y = \frac{1}{2} \dim \mathcal{M}_{\mathrm{Dol}}$.

These structures endow cohomology spaces with the following filtrations:

- weight filtration $W^\bullet$ on $H^*(\mathcal{M}_{\mathrm{B}}, \mathbb{C})$ (turning this vector space into a mixed Hodge structure);
- perverse filtration $P^\bullet$ on $H^*(\mathcal{M}_{\mathrm{Dol}}, \mathbb{Q})$.

Conjecture 1 (de Cataldo, Hausel, Migliorini [4])**.** *We have $P^\bullet = \Psi^*(W^\bullet)$ (possibly up to an affine transformation of the indices).*

Conjecture 2 (Simpson [10])**.** *There exists a smooth compactification $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$ of $\mathcal{M}_{\mathrm{B}}^{PX}$ by a simple normal crossing divisor D such that the body $|\mathcal{N}^{PX}|$ of the nerve complex $\mathcal{N}^{PX}$ of D is homotopy equivalent to S^1. Moreover, for some sufficiently large compact set $K \subset \mathcal{M}_{\mathrm{B}}^{PX}$, there exists a homotopy commutative square*

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}}^{PX} \setminus K & \xrightarrow{\Psi} & \mathcal{M}_{\mathrm{B}}^{PX} \setminus K \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \phi} \\ D^\times & \longrightarrow & |\mathcal{N}^{PX}|. \end{array}$$

Here, $D^\times \subset Y$ is a neighbourhood of ∞ in the Hitchin base.

Painlevé spaces

Let $C = \mathbb{C}P^1$, $r = 2$, d odd and assume $\sum_{j=1}^n \deg(Q_j + 1) = 4$. Then, $\dim_{\mathbb{R}} \mathcal{M}_{\mathrm{Dol}} = \dim_{\mathbb{R}} \mathcal{M}_{\mathrm{B}} = 4$. In addition to partitions of 4, at each pole p_j let us distinguish between ramified and unramified irregular types. Then, the combinatorial classification of such cases is Painlevé X for some

$$X \in \{I, II, III(D6), III(D7), III(D8), IV, V_{\mathrm{deg}}, V, VI\}.$$

The resulting moduli spaces and wild character varieties will be denoted by $\mathcal{M}_{\mathrm{Dol}}^{PX}$ and $\mathcal{M}_{\mathrm{B}}^{PX}$.

Theorem 1 (Sz. [11])**.** *Conjectures 1 and 2 hold for the spaces $\mathcal{M}_{\mathrm{Dol}}^{PX}$ and $\mathcal{M}_{\mathrm{B}}^{PX}$ (up to a shift by 1 of the indices).*

Perverse Leray filtration

Introduce

$$PH^{PX}(q,t) = \sum_{i,k} \dim_{\mathbb{Q}} \mathrm{Gr}_i^P H^k(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) q^i t^k.$$

Proposition 1. *We have*

$$PH^{PX}(q,t) = q^{-1} + (10 - \chi(F_\infty^{PX}))q^{-2}t^2 + q^{-3}t^2,$$

where $\chi(F_\infty^{PX})$ stands for the Euler-characteristic of the fiber at infinity, see Table 1.

Sketch. Abelianization procedure and sheaf counting on singular curves by Ivanics, Stipsicz, Sz. [6, 7, 8]: there exists a diagram

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{Dol}}^{PX} & \longrightarrow & E(1) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \tilde{h}} \\ \mathbb{C} & \longrightarrow & \mathbb{C}P^1. \end{array}$$

where $E(1) = \mathbb{C}P^2 \# 9 \overline{\mathbb{C}P}^2$ is the rational elliptic surface and we define

$$\tilde{h}^{-1}(\infty) = F_\infty^{PX}.$$

Geometric description of the perverse filtration due to de Cataldo and Migliorini [3] shows:

$$\begin{aligned} \mathrm{Gr}_{-3}^P H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) &= \mathbf{H}^2(Y, \mathbf{R}h_* \underline{\mathbb{Q}}) / \operatorname{Ker}(\mathbf{H}^2(Y, \mathbf{R}h_* \underline{\mathbb{Q}}) \to \mathbf{H}^2(Y_{-1}, \mathbf{R}h_* \underline{\mathbb{Q}}|_{Y_{-1}})) \\ &\cong \operatorname{Im}(\mathbf{H}^2(Y, \mathbf{R}h_* \underline{\mathbb{Q}}) \to \mathbf{H}^2(Y_{-1}, \mathbf{R}h_* \underline{\mathbb{Q}}|_{Y_{-1}})), \\ \mathrm{Gr}_{-2}^P H^2(\mathcal{M}_{\mathrm{Dol}}^{PX}, \mathbb{Q}) &= \operatorname{Ker}(\mathbf{H}^2(Y, \mathbf{R}h_* \underline{\mathbb{Q}}) \to \mathbf{H}^2(Y_{-1}, \mathbf{R}h_* \underline{\mathbb{Q}}|_{Y_{-1}})), \end{aligned}$$

where $Y_{-1} \in Y$ is a generic point. We conclude using additivity of Euler-characteristic. □

Weight filtration

By [9], for generic parameters the space $\mathcal{M}_{\mathrm{B}}^{PX}$ is a smooth affine cubic surface defined by

$$f^{PX}(x_1, x_2, x_3) = x_1 x_2 x_3 + Q^{PX}(x_1, x_2, x_3) \tag{1}$$

for some affine quadric Q^{PX}. Consider the projective compactification

$$\overline{\mathcal{M}}_{\mathrm{B}}^{PX} \subset \mathbb{C}P^3$$

defined by the homogenization of f^{PX}. Let $\widetilde{\mathcal{M}}_{\mathrm{B}}^{PX}$ be the minimal resolution of the singularities of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$, and set

$$N^{PX} = \sum_p \mu(p)$$

for the total Milnor number of the surface singularities of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$. Introduce

$$WH^{PX}(q,t) = \sum_{i,k} \dim_{\mathbb{C}} \mathrm{Gr}_{2i}^W H^k(\mathcal{M}_{\mathrm{B}}^{PX}, \mathbb{C}) q^i t^k.$$

Proposition 2. *We have*

$$WH^{PX}(q,t) = 1 + (4 - N^{PX})q^{-1}t^2 + q^{-2}t^2,$$

see Table 1.

Sketch. Follows from the spectral sequence computing the weight filtration [5], using the fact that the nerve complex $\mathcal{N}^{PX}$ is a cycle $A_{N^{PX}+2}^{(1)}$ of length $N^{PX} + 3$. □

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X	F_∞^{PX}	$PH^{PX}(q,t)$	Singularities of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$	$WH^{PX}(q,t)$
VI	$D_4^{(1)}$	$q^{-1} + 4q^{-2}t^2 + q^{-3}t^2$	$\emptyset$	$1 + 4q^{-1}t^2 + q^{-2}t^2$
V	$D_5^{(1)}$	$q^{-1} + 3q^{-2}t^2 + q^{-3}t^2$	A_1	$1 + 3q^{-1}t^2 + q^{-2}t^2$
V_{deg}	$D_6^{(1)}$	$q^{-1} + 2q^{-2}t^2 + q^{-3}t^2$	A_2	$1 + 2q^{-1}t^2 + q^{-2}t^2$
$III(D6)$	$D_6^{(1)}$	$q^{-1} + 2q^{-2}t^2 + q^{-3}t^2$	A_2	$1 + 2q^{-1}t^2 + q^{-2}t^2$
$III(D7)$	$D_7^{(1)}$	$q^{-1} + q^{-2}t^2 + q^{-3}t^2$	A_3	$1 + q^{-1}t^2 + q^{-2}t^2$
$III(D8)$	$D_8^{(1)}$	$q^{-1} + q^{-3}t^2$	A_4	$1 + q^{-2}t^2$
IV	$E_6^{(1)}$	$q^{-1} + 2q^{-2}t^2 + q^{-3}t^2$	$A_1 + A_1$	$1 + 2q^{-1}t^2 + q^{-2}t^2$
II	$E_7^{(1)}$	$q^{-1} + q^{-2}t^2 + q^{-3}t^2$	$A_1 + A_1 + A_1$	$1 + q^{-1}t^2 + q^{-2}t^2$
I	$E_8^{(1)}$	$q^{-1} + q^{-3}t^2$	$A_2 + A_1 + A_1$	$1 + q^{-2}t^2$

Table 1: Fiber at infinity of $\mathcal{M}_{\mathrm{Dol}}^{PX}$ and singularities of $\overline{\mathcal{M}}_{\mathrm{B}}^{PX}$