Middle convolution and the index of rigidity for linear differential equations with irregular singularities

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We investigate a linear algebraic formulation of the index of rigidity and the middle convolution for the linear system of differential equations which may include irregular singularities, which was introduced in arXiv:1002.2535. In particular, we show that the index of rigidity is preserved by the middle convolution in our formulation.

System of linear differential equations with irregular singularities

$$\frac{dY}{dz} = \left(-\sum_{j=1}^{m_0} A_j^{(0)} z^{j-1} + \sum_{i=1}^{r} \sum_{j=0}^{m_i} \frac{A_j^{(i)}}{(z-t_i)^{j+1}} \right) Y, \tag{1}$$

where $A_j^{(i)}$ are constant $n \times n$ matrices.

The case $m_0 = m_1 = \cdots = m_r = 0$: Fuchsian system of equations

$$\frac{dY}{dz} = \left(\frac{A^{(1)}}{z-t_1} + \cdots + \frac{A^{(r)}}{z-t_r} \right) Y. \tag{2}$$

The index of rigidity for Eq.(2) (or $\mathbf{A} = (A^{(1)}, \ldots, A^{(r)})$):

$\mathrm{idx}(\mathbf{A}) = \sum_{i=0}^{r} \dim Z(A^{(i)}) - (r-1)n^2$, $(A^{(0)} = -A^{(1)} - \cdots - A^{(r)})$.

Middle convolution for Fuchsian system

Dettweiler and Reiter [2, 3] established a linear-algebraic formulation of the middle convolution, which represents the differential equation induced by Euler's integral transformation. Define $nr \times nr$ matrices $(\overline{A}^{(1)}, \ldots, \overline{A}^{(r)})$ by

$$\overline{A}^{(1)} = \begin{pmatrix} A^{(1)} + \mu I_n & A^{(2)} & \ldots & A^{(r)} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \ldots, \overline{A}^{(r)} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ A^{(1)} & A^{(2)} & \ldots & A^{(r)} + \mu I_n \end{pmatrix}$$

$\mathcal{K}$, $\mathcal{L}(\mu)$: some explicit subspaces of $\mathbb{C}^{nr}$.

We denote the action of $\overline{A}^{(j)}$ on the quotient space $\mathbb{C}^{nr}/(\mathcal{K} + \mathcal{L}(\mu))$ by $\tilde{A}^{(j)} \in \mathrm{End}(\mathbb{C}^{nr}/(\mathcal{K} + \mathcal{L}(\mu)))$.

Middle convolution: $mc_\mu(A^{(1)}, \ldots, A^{(r)}) = (\tilde{A}^{(1)}, \ldots, \tilde{A}^{(r)})$.

Theorem 1. *(Katz [4], Dettweiler and Reiter [2, 3])* **The middle convolution for the irreducible tuple preserves the index of rigidity.**

Local conjugate about the irregular singularity $z = t$

$$\frac{dY}{dz} = \Big(\sum_{j=0}^{m} \frac{B^{[j]}}{(z-t)^{m-j+1}} + \sum_{j'=0}^{\infty} B^{[j'+m+1]}(z-t)^{j'}\Big)Y, \tag{3}$$

Set $Y = P(z)U$, $P(z) = P^{[0]} + P^{[1]}(z-t) + P^{[2]}(z-t)^2 + \dots$. Then

$$\frac{dU}{dz} = \Big(\sum_{j=0}^{m} \frac{(B')^{[j]}}{(z-t)^{m-j+1}} + \sum_{j'=0}^{\infty}(B')^{[j'+m+1]}(z-t)^{j'}\Big)U,$$

and the coefficients satisfies

$$\sum_{k'=0}^{k} \Big(P^{[k-k']}(B')^{[k']} - B^{[k']}P^{[k-k']}\Big) = 0, \quad k = 0, \dots, m. \tag{4}$$

Definition 1. *The tuple of $n \times n$ matrices $(B^{[0]}, B^{[1]}, \dots, B^{[m]})$ is conjugate to $((B')^{[0]}, (B')^{[1]}, \dots, (B')^{[m]})$, if there exist $n \times n$ matrices $P^{[0]}, P^{[1]}, \dots, P^{[m]}$ such that $P^{[0]}$ is invertible and Eq.(4) is satisfied.*

Definition 2. *Let* $V = \mathbb{C}^n$, $m \in \mathbb{Z}_{\geq 0}$, $\mathbf{B} = (B^{[0]}, B^{[1]}, \ldots, B^{[m]})$ *be a tuple of* $\mathrm{End}(V)$ *(see Eq.(3)). We define the index* $\iota(\mathbf{B})$ *by*

$$\iota(\mathbf{B}) = \dim \mathcal{C} - (m+1)(\dim V)^2,$$

where $\mathcal{C}$ *is the subspace of* $(\mathrm{End} V)^{\oplus(m+1)}$ *whose element* $(C^{[0]}, C^{[1]}, \ldots, C^{[m]})$ *satisfies* $\sum_{k'=0}^{k} \left(B^{[k-k']} C^{[k']} - C^{[k']} B^{[k-k']} \right) = 0, \quad k = 0, \ldots, m.$

The index ι is preserved by the local conjugation in Definition 1.

The index of rigidity

Definition 3. *([5]) Let* $\mathbf{A} = (A^{(0)}_{m_0}, \ldots, A^{(0)}_1, A^{(1)}_{m_1}, \ldots, A^{(1)}_0, A^{(2)}_{m_2}, \ldots, A^{(r)}_0)$ *be a tuple of* $\mathrm{End}(V)$ *(see Eq.(1)). Define the index of rigidity is by*

$$\mathrm{idx}(\mathbf{A}) = 2(\dim(V))^2 + \sum_{i=0}^{r} \iota(A^{(i)}_{m_i}, \ldots, A^{(i)}_0), (A^{(0)}_0 = -A^{(1)}_0 - \cdots - A^{(r)}_0).$$

Note that Katz [4] (also Bloch-Esnault [1]) defined the index of rigidity by $\mathrm{rig}(M) = \chi(\mathbb{P}^1, \ell_{!*}\mathcal{E}nd(M))$ for a smooth connection M on $U(\overset{\ell}{\hookrightarrow} \mathbb{P}^1)$, though I have not confirmed the coincidence of two definitions.

We define the middle convolution below. Then

Theorem 2. *([7])* **The middle convolution for the irreducible tuple preserves the index of rigidity.**

We can establish an analogue of Katz algorithm for systems of differential equation with **unramified** irregular singularities.

In particular, if the index of rigidity is two and all the irregular singularities are unramified, then the differential equations can be reduced to the scalar equation by applications of the middle convolution and the gauge transformation (addition).

Key lemma for invariance of the index of rigidity

Lemma 3. *Let* $m \in \mathbb{Z}_{\geq 0}$ *and* $\mathbf{X} = (X^{[0]}, X^{[1]}, \ldots, X^{[m]})$ *be a tuple in* $\mathrm{End}(V)$*. Then there exists a tuple* $\mathbf{B} = (B^{[0]}, B^{[1]}, \ldots, B^{[m]})$ *such that* $\mathbf{X}$ *is conjugate to* $\mathbf{B}$*,* $B^{[k]} (\in \mathrm{End}(V))$ $(k = 0, \ldots, m)$ *is written as*

$$B^{[k]} = \begin{pmatrix} B^{[k]}_{0,0} & B^{[k]}_{0,1} & \ldots & B^{[k]}_{0,k} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ B^{[k]}_{m+1,0} & B^{[k]}_{m+1,1} & \ldots & B^{[k]}_{m+1,k} & 0 \end{pmatrix},$$

and the rank of the matrix $\begin{pmatrix} B^{[0]}_{0,0} & B^{[1]}_{0,1} & \ldots & B^{[k]}_{0,k} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ B^{[0]}_{m+1,0} & B^{[1]}_{m+1,1} & \ldots & B^{[k]}_{m+1,k} & 0 \end{pmatrix}$ *is full for* $k = 0, \ldots, m$.

A description of the middle convolution for a special case

On the tuple $(B^{[0]}, B^{[1]}, B^{[2]})(\sim B^{[0]}/z^3 + B^{[1]}/z^2 + B^{[2]}/z)$ such that

$$B^{[0]}= \begin{pmatrix} B^{[0]}_{0,0} & 0 & 0 & 0 \\ B^{[0]}_{1,0} & 0 & 0 & 0 \\ B^{[0]}_{2,0} & 0 & 0 & 0 \\ B^{[0]}_{3,0} & 0 & 0 & 0 \end{pmatrix}, B^{[1]}= \begin{pmatrix} B^{[1]}_{0,0} & B^{[1]}_{0,1} & 0 & 0 \\ B^{[1]}_{1,0} & B^{[1]}_{1,1} & 0 & 0 \\ B^{[1]}_{2,0} & B^{[1]}_{2,1} & 0 & 0 \\ B^{[1]}_{3,0} & B^{[1]}_{3,1} & 0 & 0 \end{pmatrix}, B^{[2]}= \begin{pmatrix} B^{[2]}_{0,0} & B^{[2]}_{0,1} & B^{[2]}_{0,2} & 0 \\ B^{[2]}_{1,0} & B^{[2]}_{1,1} & B^{[2]}_{1,2} & 0 \\ B^{[2]}_{2,0} & B^{[2]}_{2,1} & B^{[2]}_{2,2} & 0 \\ B^{[2]}_{3,0} & B^{[2]}_{3,1} & B^{[2]}_{3,2} & 0 \end{pmatrix},$$

with the full rank condition in Lemma 3, the middle convolution is calculated as

$$\tilde{B}^{[0]} \sim \begin{pmatrix} B^{[0]}_{0,0} & B^{[1]}_{0,1} & B^{[2]}_{0,2} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{B}^{[1]} \sim \begin{pmatrix} B^{[1]}_{0,0} & B^{[2]}_{0,1} & * & * \\ B^{[0]}_{1,0} & B^{[1]}_{1,1} & B^{[2]}_{1,2} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{B}^{[2]} \sim \begin{pmatrix} B^{[2]}_{0,0} + 3\mu & * & * & * \\ B^{[1]}_{1,0} & B^{[2]}_{1,1} + 2\mu & * & * \\ B^{[0]}_{2,0} & B^{[1]}_{2,1} & B^{[2]}_{2,2} + \mu & * \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A description of the index for the tuple $(B^{[0]}, B^{[1]}, B^{[2]})$

We consider the tuple $(C^{[0]}, C^{[1]}, C^{[2]})$ in Definition 2.

Relations: $B^{[0]}C^{[0]} = C^{[0]}B^{[0]}$, $B^{[1]}C^{[0]} + B^{[0]}C^{[1]} = C^{[1]}B^{[0]} + C^{[0]}B^{[1]}$,

$$B^{[2]}C^{[0]} + B^{[1]}C^{[1]} + B^{[0]}C^{[2]} = C^{[2]}B^{[0]} + C^{[1]}B^{[1]} + C^{[0]}B^{[2]}.$$

Write $C^{[k]} = \begin{pmatrix} C^{[k]}_{0,0} & C^{[k]}_{0,1} & C^{[k]}_{0,2} & C^{[k]}_{0,3} \\ C^{[k]}_{1,0} & C^{[k]}_{1,1} & C^{[k]}_{1,2} & C^{[k]}_{1,3} \\ C^{[k]}_{2,0} & C^{[k]}_{2,1} & C^{[k]}_{2,2} & C^{[k]}_{2,3} \\ C^{[k]}_{3,0} & C^{[k]}_{3,1} & C^{[k]}_{3,2} & C^{[k]}_{3,3} \end{pmatrix}$ for $k = 0, 1, 2$. Then we have $C^{[0]}_{i,j} = 0$ $_{(0 \leq i < j \leq 3)}$, $C^{[1]}_{i,j} = 0$ $_{(i,j)=(0,2),(0,3),(1,3)}$ and $C^{[2]}_{0,3} = 0$.

The elements of $C^{[k]}_{i,j}$ for $j > i + k$ or $i + k > 2$ are determined by the elements of $C^{[k]}_{i,j}$ for $j \leq i + k \leq 2$.

Therefore the system of the equations for $C^{[k]}_{i,j}$ ($j \leq i+k \leq 2$) is essential.

Comparison with the tuple $(\tilde{C}^{[0]}, \tilde{C}^{[1]}, \tilde{C}^{[2]})$ for $(\tilde{B}^{[0]}, \tilde{B}^{[1]}, \tilde{B}^{[2]})$ is a key step of the proof of Theorem 2.

Definition of middle convolution

$V=\mathbb{C}^n,\mu\in\mathbb{C},V_j^{(i)}=V\ (i=0,\dots,r,\ j=\delta_{i,0},\dots m_i),$

$V^{(0)}=V_{m_0}^{(0)}\oplus V_{m_0-1}^{(0)}\oplus\dots\oplus V_1^{(0)},\ V^{(i)}=V_{m_i}^{(i)}\oplus\dots\oplus V_1^{(i)}\oplus V_0^{(i)}\ (i\ge 1),$

$V'=V^{(0)}\oplus V^{(1)}\oplus\cdots\oplus V^{(r)}(=V^{\oplus M}),\ (M=r+m_0+m_1+\cdots+m_r).$

$$\overline{A}_j^{(0)}=\begin{pmatrix} \left.\begin{matrix}\mu I_n & & \\ & \ddots & \\ & & \mu I_n\end{matrix}\right\}{\scriptstyle m_0-j} & & & & & & & & \\ A_{m_0}^{(0)} & \cdots & A_2^{(0)} & A_1^{(0)} & A_{m_1}^{(1)} & \cdots & A_0^{(1)} & A_{m_2}^{(2)} & \cdots & A_0^{(r)} \\ & & & & & & & & & \\ & & & & & & & & & \end{pmatrix},$$

$$
\overline{A}_j^{(i)} = \left(\begin{array}{c}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} m_0+(m_1+1)+\cdots+(m_{i-1}+1) \\
\left.\begin{array}{ccc} \mu I_n & & \\ & \ddots & \\ & & \mu I_n \end{array}\right\} m_i - j \\
A_{m_0}^{(0)} \cdots A_{\delta_{i,0}}^{(i-1)} \; A_{m_i}^{(i)} \; \cdots \; A_1^{(i)} \; A_0^{(i)} + \mu I_n \, A_{m_{i+1}}^{(i+1)} \cdots A_0^{(r)} \\
\\
\\
\end{array}\right).
$$

These matrices represent the transformation of the differential equation by Euler's integral transformation of solutions (see [6]). Irreducibility of the tuple is not inherited. The middle convolution is defined by taking a suitable quotient of the matrices. Set $A_0^{(0)} = -A_0^{(1)} - \cdots - A_0^{(r)}$.

We define the subspaces $\mathcal{K} = \bigoplus_{i=1}^{r} \mathcal{K}^{(i)}$ and $\mathcal{L}(\mu)$ $(\mu \neq 0)$ by

$$\mathcal{K}^{(i)} = \left\{ \begin{pmatrix} v_{m_i}^{(i)} \\ v_{m_i-1}^{(i)} \\ \vdots \\ v_0^{(i)} \end{pmatrix} \in V^{(i)} \,\middle|\, \begin{pmatrix} A_{m_i}^{(i)} & A_{m_i-1}^{(i)} & \cdots & A_0^{(i)} \\ 0 & A_{m_i}^{(i)} & \cdots & A_1^{(i)} \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & A_{m_i}^{(i)} \end{pmatrix} \begin{pmatrix} v_{m_i}^{(i)} \\ v_{m_i-1}^{(i)} \\ \vdots \\ v_0^{(i)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\},$$

$$\mathcal{L}(\mu) = \left\{ v \in V' \,\middle|\, \begin{array}{c} v_j^{(i)} = 0,\ (i \neq 0, j \neq 0), \quad v_0^{(1)} = \cdots = v_0^{(r)} =: -\ell, \\ \begin{pmatrix} A_{m_0}^{(0)} & \cdots & A_1^{(0)} & A_0^{(0)} - \mu I_n \\ 0 & A_{m_0}^{(0)} & \cdots & A_1^{(0)} \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & A_{m_0}^{(0)} \end{pmatrix} \begin{pmatrix} v_{m_0}^{(0)} \\ \vdots \\ v_1^{(0)} \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \end{array} \right\}.$$

We denote the tuple induced by $\overline{\mathbf{A}}$ in End$(V'/(\mathcal{K} + \mathcal{L}(\mu)))$ by $mc_\mu(\mathbf{A}) = (\tilde{A}_{m_0}^{(0)}, \ldots, \tilde{A}_1^{(0)}, \tilde{A}_{m_1}^{(1)}, \ldots, \tilde{A}_0^{(r)})$. Set also $mc_\mu(V) = V'/(\mathcal{K} + \mathcal{L}(\mu))$ and call them the middle convolution with the parameter μ.

Theorem 4. *([7], c.f. [2, Corollary 3.6]) If an $\mathbf{A}$-module V is irreducible, then $mc_\mu(V)$ is irreducible or the zero module for all $\mu \in \mathbb{C}$.*

Theorem 5. *([7], c.f. [2, Theorem 3.5]) If V is irreducible $\mathbf{A}$-module, then the $\mathbf{A}$-module $mc_{\mu_2}(mc_{\mu_1}(V))$ is isomorphic to $mc_{\mu_1+\mu_2}(V)$ for all $\mu_1, \mu_2 \in \mathbb{C}$. In particular the modules $mc_{-\mu}(mc_\mu(V))$ is isomorphic to V for all $\mu \in \mathbb{C}$.*

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