An Analogue of Miyaoka-Yau type inequality for certain threefold with regards to the associated third Chen classes.

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1 Main purpose

For an extremal curve nbd (X, C) of type (IIA),

(1) to induce an analogue of Miyaoka-Yau type inequality as "$(-K_X)^3$(or,$c_1(-K_X)^3$) $\leq a\, c_2(-K_X)^2 \tilde{\otimes} \mathscr{O}_{X^\sharp}(H^\sharp) + b\, c_3(-K_X)$ ($\exists a, b \in \mathbb{Z}_{\geq 0}$,$H^\sharp \in |\mathscr{O}_{X^\sharp}|$)",with

(2) regarding to c_3 about the followings: i) the divisorial contraction associated to (X, C) and a pencil for $H \in |\mathscr{O}_X|_C$ (the permissible linear pencil), and ii) a characterization of a $\mathbb{Q}$-conic bundle by the above i).

2 Several observations

For (X, C) of type (IIA),the followings are observed:

2.1 *l*-splitness of $\mathrm{gr}^2(\mathscr{O}, J)$

(i) ([3, III]) Let $f : (X, C) \to (Z, o)$ be the corresponding contraction. Assume that $g := f|_D : D \to Z$ is finite for $D \in |-K_X|$.Then,we have $f_*\omega_X(D) \to g_*\omega_D \to \omega_D \to 0$ (exact) by $0 \to \omega_X \to \omega_X(D) \to \omega_D \to 0$ (exact), and $\exists$ a surjection $f_*\omega_X(D) \twoheadrightarrow g_*\omega_D/\omega_D$.By this,we have a surejction $gr^0_C\mathscr{O}_C \twoheadrightarrow gr^0_C\omega_X$.

(ii) ([4, I,Lem.(3.5)]) Let J be a C-laminal ideal with $d = 2$. Suppose $gr^2(\mathscr{O}, J) = (aP) \oplus (-1 + 3P)(a \in \mathbb{Z}_{\geq 0}, P \in C)$. Then,$\exists$ a global section s of $F^2(\mathscr{O}(D), J)$ ($D \in |-K_X|$) such that $E := \{s = 0\} \in |-K_X|_C$ induces an l-isomorphism $gr^2(\mathscr{O}, J) = (aP)\tilde{\oplus}\mathscr{O}_C(-E)$.

2.2 Properties of the pencil induced by $\omega_X^\vee$

([4, Thm.(1.1)]) For f as the above (2.1.(i)), assume that (X, C) has no flipping. Then, assuming general $H \in |\mathscr{O}_X|_C$ is normal, the followings a)-c) occur: a) f is divisorial,$f(H) \ni o$ is of type A_1; b) f is divisorial,$f(H) \ni o$ is of type D_5; c) f is a $\mathbb{Q}$-conic bundle.

3 An Analogue of Miyaoka-Yau type inequality for type (IIA)

3.1 the associated (orbifold) Chenn classes

For (X, C),let J be a C-laminal ideal of pure width d, $\mathscr{S}$ be a coherent $\mathscr{O}_X$-module, $d \geq 2$,$q := [n/d]$,and $r := n - qd$.

i) ([2, (8.2)]) there exists a natural homomorphism $\gamma^n(\mathscr{S}, J) : S^q(gr^d(\mathscr{O}, J)) \otimes (gr(\mathscr{O}, J))^{\otimes r} \otimes gr^0_C\mathscr{S} \to gr^n(\mathscr{S}, J)$, which induces a 0-sequence $E(J)$: $0 \to gr^1(\mathscr{O}, J)^{\otimes d} \xrightarrow{\Delta_J} gr^0_C(J) \xrightarrow{\alpha_J} \mathrm{Ker}[gr^1_c\mathscr{O} \xrightarrow{\beta_J} gr^1(\mathscr{O}, J)] \to 0$.

ii) ([2, (8.6)]) $E(J)$ induces a saturated filtration $gr^n(\mathscr{S}, J) = \Phi^0 gr^n(\mathscr{S}, J) \supset \cdots \supset \Phi^i gr^n(\mathscr{S}, J) \supset \cdots \Phi^{q+1} gr^n(\mathscr{S}, J) = 0$ such that $gr^{n,i}(\mathscr{S}, J) = \Phi^i gr^n(\mathscr{S}, J)/\Phi^{i+1} gr^n(\mathscr{S}, J)$ is a torsion free $\mathscr{O}_C$-module of rank ρ for each $i \in [0, q]$,and there exists a natural induced map ($i \in [0, q]$) $\gamma^{n,i}(\mathscr{S}, J) : gr^1(\mathscr{O}, J)^{\otimes(di+r)} \otimes (\mathrm{Im}\,\alpha_J)^{\otimes(q-i)} \otimes gr^0_C\mathscr{S} \to gr^{n,i}(\mathscr{S}, J)$.

Proposition-Definition 3.1 *For an extremal curve nbd (X, C) of type (IIA), its associated Chern classes $c_a := c_a(X^\sharp, C^\sharp) = c_a^{\mathrm{orb}}(X, C)$ ($a \in [1, 3]$) of (X, C),as the orbifold Chern classes ([1]),are given by $\mathscr{E} := \mathrm{gr}^n(\mathscr{O}, J)$ with $\gamma^{n,i}(\mathscr{O}, J)$ (3.1.ii)) for (X, C) ([2, (8.11.1,ii)]) as follows,where $(X^\sharp, C^\sharp)$ is the canonical cover of (X, C):*

- $c_3(\mathscr{E}^\sharp) = \mathrm{ldeg}(\mathscr{D}_P)$, *where* $\mathscr{D}_P := \mathscr{O}_{C^\sharp}\bar{f}^{-1}$,
- $c_2(\mathscr{E}^\sharp) = \Sigma_{i=0}^q(q-i)\cdot\mathrm{ldeg}(\mathscr{L}^\sharp) + \Sigma_{i=0}^q(di+r)\cdot\mathrm{ldeg}(\mathscr{M}^\sharp)$, *where* $\mathscr{L}^\sharp = \mathscr{O}_{C^\sharp}\cdot e$,*or* $\mathscr{O}_{C^\sharp}\cdot s_2$ *and* $\mathscr{M}^\sharp = \mathscr{O}_{C^\sharp}\cdot s_1$,
- $c_1(\mathscr{E}^\sharp) = \mathrm{ldeg}(gr^0_{C^\sharp}\mathscr{O}^\sharp)$.□

Under the conditions of (2.1.(ii)),we have Props (3.1) and (3.2) as follows:

Proposition 3.1 *Let $\mathscr{E} := gr^2(\mathscr{O}, J) = (aP)\tilde{\oplus}\mathscr{O}_C(-E)$, and $E^\sharp$ be the image of E by the canonical cover.Then,we have $(E^\sharp \cdot det(\mathscr{E}^\sharp))_{C^\sharp} \leq max[c_2(\mathscr{E}^\sharp) + c_3(\mathscr{E}^\sharp), 0]$.*

Proposition 3.2 *For $\mathbb{P}^\sharp := \mathbb{P}(gr^2(\mathscr{O}, J)^\sharp) \xrightarrow{c^\sharp} \mathbb{P} := \mathbb{P}(gr^2(\mathscr{O}, J)) \xrightarrow{\pi} (X, C)$, and $\pi^\sharp := \pi \circ c^\sharp$,where $c^\sharp$ is the induced map from the canonical cover. then we have $\chi(\mathbb{P}^\sharp, S^{3t}(\mathscr{E}^\sharp)\tilde{\otimes}\mathscr{O}_{\mathbb{P}^\sharp}(-t\, c_1(det((\pi^\sharp)^*(\mathscr{E}^\sharp))))) = \chi(X, S^{3t}(\mathscr{E}) \otimes \mathscr{O}_X(-t\, c_1(det(\mathscr{E})))) \sim (t^4/4!)(E - det(\mathscr{E}))^4 + (t^3/3!)(E - det(\mathscr{E}))^3 + O(t^2)$.* □

Since $S^3(gr^0_C\Omega^{1\,\vee}_X) \hookrightarrow gr^0_C\mathscr{O}$ ([2]) and $gr^0_C\mathscr{O} \twoheadrightarrow gr^0_C\omega_X$ (2.1.(i)), by (2.1.(ii)) and (3.1-2),we have

Proposition 3.3 *(Bogomolov-Gieseker type inequality) For (X, C) of type (IIA) with $\mathscr{E} := gr^2(\mathscr{O}, J)$,then we have $c_1(S^3(gr^0_C\Omega^{1\,\vee}_X)) \leq a\, c_2(\mathscr{E}^\sharp)\tilde{\otimes}\mathscr{O}_{X^\sharp}(H^\sharp) + b\, c_3(\mathscr{E}^\sharp)$ ($\exists a, b \in \mathbb{Z}_{\geq 0}$,a general $H \in |\mathscr{O}_X|$).*

3.2 A Characterization of $\mathbb{Q}$-conic bundles

Proposition 3.4 *(for (1.2.ii),details omitted) For (3.3),we have ([4]): 1) If $c_3(\mathscr{E}^\sharp) \ni O$ (as a cycle),then $3 \leq c_3(\mathscr{E}^\sharp) \leq 4$ and $r > 0$ or $s > 0$ (f is a del Pezzo fibration). 2) Otherwise,then $c_3(\mathscr{E}^\sharp) = 2$ and $r = s = 0$ (f is a $\mathbb{Q}$-conic bundle).*

References

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