Noncommutative Instantons and Reciprocity

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D-Modules, quantum geometry and related topics

Based on

MH& Toshio Nakatsu (Setsunan U), (to appear)

Noncommutative Instantons and Reciprocity,

cf. (Proceedings:) ADHM construction of noncommutative instantons [arXiv:1311.5227]

1. Introduction

- **Non-Commutative (NC) spaces are defined by noncommutativity of spatial coordinates:**

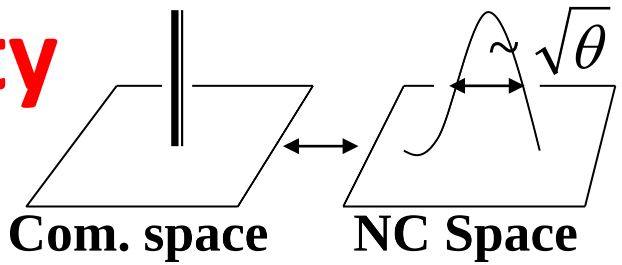
$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$ $\theta^{\mu\nu}$: NC parameter (real const.)

(cf. CCR in QM : $[q, p] = i$)

(→ ``space-space uncertainty relation'' →)

Resolution of singularity

(→ new physical objects)

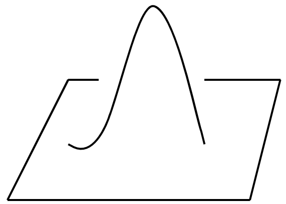


Ex) Resolution of small instanton singularity

(→ U(1) instantons) [Nekrasov-Schwarz]

Memo on instantons

- Instanton = instant + on ＝瞬間子



localized configuration in 4-dim.

- Descendant of the Twistor correspondence

  Anti-Self-Dual connection

  Holomorphic vector bdl.

- Instantons play important roles in geometry and physics. e.g. Donaldson invariant, Nekrasov partition fcn., AGT corresp. …

ASDYM eq. in 4-dim. with G=U(N)

- **ASDYM eq. (real rep.)** $\mu,\nu = 1,2,3,4$

$$F_{12} = -F_{34}, \quad F_{13} = -F_{42}, \quad F_{14} = -F_{23}.$$

$$F_{\mu\nu}(x) := \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu$$

Curvature=Field strength

$A_\mu(x)$: **Connection=Gauge field (N × N anti-Hermitian)**

$$(\Leftrightarrow F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1 z_2} = 0 \quad (cpx.\, rep.))$$

- **There are two descripitions of NC extension:**
  - **Moyal-product formalism** (deformation quantization)
  - **Operator formalism** (Connes' theory)

NC ASDYM eq. with G=U(N) in Moyal

- **NC ASDYM eq. (real rep.)** $\mu,\nu=1,2,3,4$

$$F_{12}^{*}=-F_{34}^{*},\qquad (F_{\mu\nu}^{*}:=\partial_\mu A_\nu-\partial_\nu A_\mu+A_\mu * A_\nu-A_\nu * A_\mu)$$

$$F_{13}^{*}=-F_{24}^{*},$$

$$F_{14}^{*}=-F_{23}^{*}$$

$$\theta^{\mu\nu}=\left[\begin{array}{cc|cc} 0 & \theta^1 & \multicolumn{2}{c}{\multirow{2}{*}{O}} \\ -\theta^1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{O}} & 0 & \theta^2 \\ & & -\theta^2 & 0 \end{array}\right]$$

(Spell:All products are Moyal products.)

$$f(x)*g(x):=f(x)\exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right)g(x)$$

$$=f(x)\cdot g(x)+i\frac{\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x)+O(\theta^2)$$

Under the spell, we can calculate :

$$[x^\mu,x^\nu]_*:=\cancel{x^\mu\cdot x^\nu}+\frac{i}{2}\theta^{\mu\nu}-(\cancel{x^\nu\cdot x^\mu}-\frac{i}{2}\theta^{\mu\nu})$$

$$=i\theta^{\mu\nu}$$

:NC Space!

Under the spell, the solution is deformed:

commutative one

$$A(x,\theta)=A^{(0)}(x)+\theta A^{(1)}(x)+\theta^2A^{(2)}(x)+\cdots,$$

2. Atiyah-Drinfeld-Hitchin-Manin Construction

based on duality for the instanton moduli space

4dim. ASDYang-Mills eq.
(Difficult)

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

N × N PDE

Sol.= instantons
(G=U(N), C_2 =k)

$$A_\mu : N \times N$$

Gauge trf.:

$$A_\mu \mapsto g^{-1} A_\mu g + g^{-1} \partial_\mu g$$
$$g \in U(N)$$

1:1

ADHM eq. (≒0dim. ASDYM)
(Easy)

$$[B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = 0$$
$$[B_1, B_2] + IJ = 0$$

k × k Matrix eqs.

Sol.=ADHM data
(G=`U(k)')

$$B_{1,2} : k \times k, \quad I : k \times N, \quad J : N \times k$$

Gauge trf.:

$$B_{1,2} \mapsto \tilde{g}^{-1} B_{1,2} \tilde{g}, \quad \tilde{g} \in U(k)$$
$$I \mapsto \tilde{g}^{-1} I, \quad J \mapsto J \tilde{g}$$

Fourier-Mukai-Nahm transformation

Beautiful reciprocity between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq. on a 4-torus

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

N × N PDE

Dirac eq.

$$\xrightarrow{\quad G \quad}$$
$$\xleftarrow[\quad F \quad]{}$$

Dirac eq.

4dim. ASD Yang-Mills eq. on the dual torus

$$\widetilde{F}_{\xi_1\bar{\xi}_1} + \widetilde{F}_{\xi_2\bar{\xi}_2} = 0$$
$$\widetilde{F}_{\xi_1\xi_2} = 0$$

k × k PDE

Sol.=instantons
($G=U(N)$, C_2 =k)

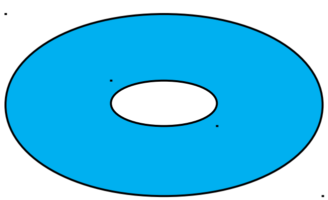
1:1
(reciprocity)

⇕

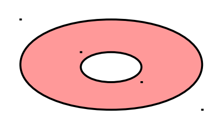
Sol.=the dual instantons
($G=U(k)$, C_2=N)

$$A_\mu : N \times N$$

$$\widetilde{A}_\mu : k \times k$$



Define the maps F & G,
& G∘F=id. & F∘G=id.



On a 4-torus

On the dual 4-torus

Fourier-Mukai-Nahm transformation

Beautiful duality between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq. on a 4-torus

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

4dim. ASD Yang-Mills eq. on the dual torus

$$\widetilde{F}_{\xi_1\bar{\xi}_1} + \widetilde{F}_{\xi_2\bar{\xi}_2} = 0$$

$$\widetilde{F}_{\xi_1\xi_2} = 0$$

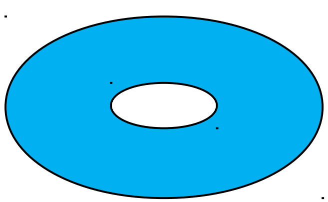
k × k PDE

1:1

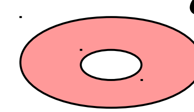
Sol.=instantons (G=U(N), C_2 =k) $\longleftrightarrow$ Sol.=the dual instantons (G=U(k), C_2=N)

$$A_\mu(x) = \left\langle V, \partial_\mu V \right\rangle_\xi \xleftarrow{\text{map F (Dirac eq.)}} \widetilde{A}_\mu(\xi) : k \times k$$

N×N



$$\nabla^+ V = \bar{e}^\mu \otimes \left(\frac{\partial}{\partial \xi^\mu} + \widetilde{A}_\mu - i x_\mu\right) V = 0 \qquad \bar{e}^\mu := (i\sigma_a, 1_2)$$



$$V : 2k \times \underline{N}$$

On a 4-torus $: x_\mu$

Family index thm.

On the dual 4-torus $: \xi_\mu$

Fourier-Mukai-Nahm transformation

Beautiful duality between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq. on a 4-torus

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

N × N PDE

4dim. ASD Yang-Mills eq. on the dual torus

$$\widetilde{F}_{\xi_1\bar{\xi}_1} + \widetilde{F}_{\xi_2\bar{\xi}_2} = 0$$
$$\widetilde{F}_{\xi_1\xi_2} = 0$$

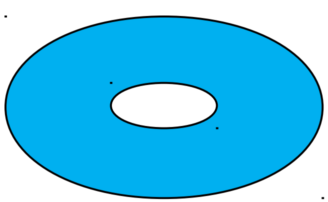
k × k PDE

1:1

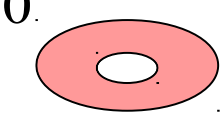
Sol.=instantons (G=U(N), C_2 =k) ⟷ Sol.=the dual instantons (G=U(k), C_2=N)

$$A_\mu(x) : N \times N \xrightarrow{\text{map G (Dirac eq.)}} \widetilde{A}_\mu(\xi) = \left\langle \psi, \widetilde{\partial}_\mu \psi \right\rangle_\xi$$

k×k



$$\bar{e}_\mu D_\mu \psi = \bar{e}^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + A_\mu - i\xi_\mu\right)\psi = 0$$



$$\psi : 2N \times \underline{k}$$

Family index thm.

On a 4-torus $: x_\mu$

On the dual 4-torus $: \xi_\mu$

Fourier-Mukai-Nahm trf. (radii of the torus→∞)

reciprocity between instanton moduli on $\mathbf{R}^4$ and instanton moduli on ``1pt.'' [cf. van Baal, hep-th/9512223]

4dim. ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

N × N PDE

0dim. ASD Yang-Mills eq.

$$\widetilde{F}_{\mu\nu} := \cancel{\partial_\mu}\widetilde{A}_\nu - \cancel{\partial_\nu}\widetilde{A}_\mu + [\widetilde{A}_\mu, \widetilde{A}_\nu]$$

$$\widetilde{F}_{\xi_1\bar{\xi}_1} + \widetilde{F}_{\xi_2\bar{\xi}_2} = 0$$
$$\widetilde{F}_{\xi_1\xi_2} = 0$$

Matrix eq. !

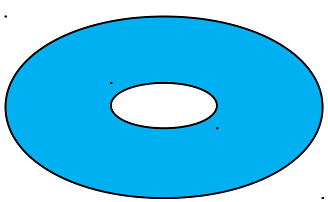
k × k ~~PDE~~

1:1

Sol.=instantons (G=U(N), C_2 =k) ⟷ Sol.=``dual instantons'' (G=U(k), ``C_2=N'')

map F (0dim Dirac eq.)

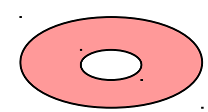
$$A_\mu = V^+ \partial_\mu V \longleftarrow \qquad \widetilde{A}_\mu : k \times k$$



$$\nabla^+ V = \bar{e}^\mu \otimes \left(\cancel{\frac{\partial}{\partial \xi^\mu}} + \widetilde{A}_\mu - i x_\mu\right) V = 0$$

Matrix eq. !

$$V : 2k \times \underline{N}$$



On a 4-torus→$\mathbf{R}^4$

Linear alg.

On the dual 4-torus→1 pt.

Other limits and related works

- 3radii→∞&1radius→0 : [Nahm, Hitchin, Nakajima,…]

  monopole on 3-dim ←→ Nahm data on 1-dim
- 2radii→∞&2radii→0 :

  Hitchin system on 2dim ←→ Hitchin system 2dim
- 3radii→∞&finite1radius : [Hurtubise-Murray,…]

  caloron on R^2x S^1 ←→ Nahm data on S^1
- 2radii→∞&finite2radii : : [Jardim, Mochizuki, …]

  instanton on R^2x T^2 ←→ Hitchin system on T^2
- 1radii→∞&finite3radii : : [Charbonneau, Yoshino,…]

  instanton on R x T^3 ←→ monopole on T^3

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) Commutative BPST instanton（N=2, k=1）

4dim. ASDYang-Mills eq.

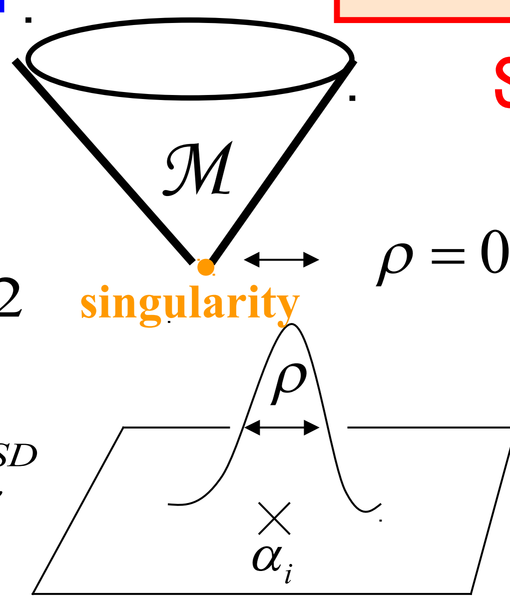
$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$
$$F_{z_1 z_2} = 0$$

N × N PDE

ADHM eq. (≒0dim. ASDYM)

$$\mu_R = [B_1, B_1^+] + [B_2, B_2^+] + I I^+ - J^+ J = 0$$
$$\mu_C = [B_1, B_2] + IJ = 0$$

k × k matrix eq.



BPST instanton
(G=U(2), C_2 =1)

$$A_\mu = \frac{i(x-b)^\nu \eta_{\mu\nu}^{ASD}}{(z-\alpha)^2 + \rho^2}, \quad 2\times 2$$

$$F_{\mu\nu} = \frac{2i\rho^2}{((z-\alpha)^2+\rho^2)^2}\eta_{\mu\nu}^{ASD}$$

$\rho \to 0$ ：singular

Sol.=ADHM data
(G=`U(1)’)

position

$$B_{1,2} = \alpha_{1,2}, \quad 1\times 1$$

$$I = (\rho, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

size

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction

Ex.) NC BPST instanton（N=2, k=1）

NC ASDYang-Mills eq.

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$$

$$F_{z_1 z_2} = 0$$

N × N PDE

NC ADHM eq.

$$\mu_R = [B_1, B_1^+] + [B_2, B_2^+] + I\,I^+ - J^+ J = \zeta$$

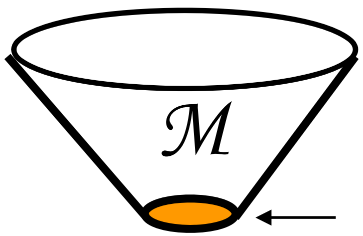
$$\mu_C = [B_1, B_2] + IJ = 0$$

k × k marix eq.

NC BPST instanton
(G=U(2), C_2 =1)

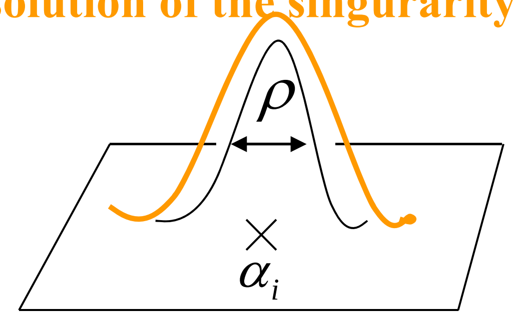
By calculation of TrF∧F

$A_\mu, F_{\mu\nu}$: exact sol.



$\rho=0$

Resolution of the singurarity!



$\rho \to 0$: regular!

Sol.：ADHM data
(G=`U(1)’)

position

$B_{1,2} = \alpha_{1,2},$

$$I = (\sqrt{\rho^2 + \zeta}, 0),\ J = \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

size　Fat by ζ !

Do k × k ADHM data give
Instanton number k
in general ? (We prove this.)

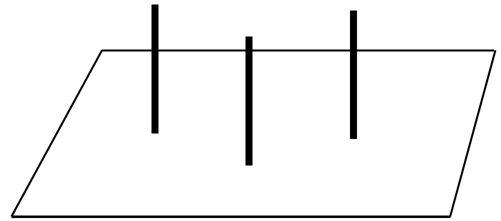
The instanton moduli space

- **in commutative spaces**

$$\overline{\mathrm{M}}_{k,N} = \mathrm{M}_{k,N} \cup (\mathrm{M}_{k-1,N} \times R^4) \cup (\mathrm{M}_{k-2,N} \times Sym^2 R^4) \cup \ldots$$
$$\cup (\mathrm{M}_{1,N} \times Sym^{k-1} R^4) \cup Sym^k R^4$$

k size-zero instantons

Symmetric Product



- **In noncommutative spaces**

$$\overline{\mathrm{M}}_{k,N} = \mathrm{M}_{k,N} \cup (\mathrm{M}_{k-1,N} \times R^4) \cup (\mathrm{M}_{k-2,N} \times \widetilde{Sym^2 R^4}) \cup \ldots$$
$$\cup (\mathrm{M}_{1,N} \times \widetilde{Sym^{k-1} R^4}) \cup \widetilde{Sym^k R^4}$$

k U(1) instantons

Hilbert Scheme

- **dim** $\mathrm{M}_{k,N} = 2 \cdot 2k^2 + 2 \cdot 2Nk - 3k^2 - k^2 = 4Nk$ **算数**

3. Proof of the reciprocity:(inst)↔(ADHM)

$$\text{NC instanton} \underset{F}{\overset{G}{\rightleftarrows}} \text{NC ADHM}$$

$A_\mu : N \times N$

$B_{1,2} : k \times k,$

$I : k \times N, \quad J : N \times k$

(i) ASD (ASDYM eq.)
(ii) C_2=k
~~(iii) $\mathfrak{D}$^2 has inverse~~

(i) ASD (ADHM eq.)
(ii) matrix size = k, N
~~(iii) ∇^2 has inverse~~

(iii) is automatically satisfied in the noncommutative situation
[Maeda-Sako] [Nakajima] (For any θ [MH, Nakatsu])

Proof of the one-to-one ⇔ Define the maps F & G,
& G∘F=id. & F∘G=id.

F:(ADHM)→(inst):ADHM construction

NC instanton $\xleftarrow[\text{Dirac eq.}]{\text{0dim.}}$ NC ADHM

$$\nabla^+ * V = 0, \quad V^+ * V = 1_N$$

$$A_\mu = V^+ * \partial_\mu V : N \times N$$

$$\nabla = \begin{pmatrix} I^+ & J \\ \bar{z}_2 - B_2^+ & -(z_1 - B_1) \\ \bar{z}_1 - B_1^+ & z_2 - B_2 \end{pmatrix}$$

$0\,\mathrm{dim}\ Dirac\ op.(N+2k)\times 2k$

$B_{1,2} : k \times k,$

$I : k \times N, \quad J : N \times k$

(i) ASD(ASDYM eq) [Nekasov-Schwarz]

(ii) C_2=k ←**[MH Nakatsu],…**

(i) ASD (ADHM eq.)

(ii) matrix size= k, N

We prove the NC version of the formula: **cf.[Atiyah, Hori]**

$$\int d^4x\, Tr_N F^*_{\mu\nu} * F^*_{\mu\nu} = -\int d^4x\, Tr_{2k} \Omega^*_{\mu\nu} * \Omega^*_{\mu\nu} \qquad cf. \quad ch(E \oplus F) = ch(E) + ch(F)$$

$$where \quad \Omega_{\mu\nu} := \partial_\mu \omega - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]_*$$

$$\underset{2k\times 2k}{\omega_\mu := \tilde{\nabla}^+ * \partial_\mu \tilde{\nabla}}, \qquad \underset{(N+2k)\times 2k}{\tilde{\nabla} := \nabla * (\nabla^+ * \nabla)^{1/2}}$$

F : (ADHM)→(inst):ADHM construction

NC instanton ← (0dim. Dirac eq.) — NC ADHM

$$\nabla^+ * V = 0, \quad V^+ * V = 1_N$$

$$A_\mu = V^+ * \partial_\mu V : N \times N$$

$$\nabla = \begin{pmatrix} I^+ & J \\ \bar{z}_2 - B_2^+ & -(z_1 - B_1) \\ \bar{z}_1 - B_1^+ & z_2 - B_2 \end{pmatrix}$$

$0\,\mathrm{dim}\, Dirac\ op. (N+2k)\times 2k$

$$B_{1,2} : k \times k,$$

$$I : k \times N, \quad J : N \times k$$

Then:

$$C_2 := \frac{1}{16\pi^2}\int d^4x\, Tr_N F^*_{\mu\nu} * F^*_{\mu\nu} = \frac{-1}{16\pi^2}\int d^4x\, Tr_{2k}\Omega^*_{\mu\nu} * \Omega^*_{\mu\nu}$$

$$= \frac{1}{24\pi^2}\int_{S^3} \underline{Tr_k 1_k} \cdot Tr_2 (g^{-1}dg)^3 = k, \qquad g := \frac{x^\mu e_\mu}{r}$$

comes from the size of the ADHM data!

G : (inst)→(ADHM): inverse construction

NC instanton $\xrightarrow[\text{Dirac eq.}]{\text{4dim.}}$ NC ADHM

$$\bar{e}_\mu D_\mu * \psi = 0, \quad \int d^4 x \psi^+ * \psi = 1_k$$

$A_\mu : N \times N$ $\qquad e^\mu D_\mu : 4\,\mathrm{dim}\ Dirac\ op.$

$$B_{1,2} = \int d^4 x\, z_{1,2} * \psi^+ * \psi : k \times k,$$

$$\psi \approx \frac{I^+, J}{r^3} : N \times k$$

(i) ASD (ASDYM eq.)
(ii) C_2=k

(i) ASD (ADHM eq.)
(ii) matrix size= k, N

[Maeda-Sako2009] proves existence of the Dirac zero-mode by a formal power expansion of θ, recursively.
cf. [Nekrasov. K.Kim, B.H.Lee, H.Yang]

commutative input

$$\psi(x,\theta) = \psi^{(0)} + \theta\psi^{(1)} + \theta^2\psi^{(2)} + \cdots,$$
$$A(x,\theta) = A^{(0)} + \theta A^{(1)} + \theta^2 A^{(2)} + \cdots,$$

$G \circ F = id$: (ADHM)→(inst)→(ADHM)

NC ADHM $\xrightarrow[\text{Dirac eq.}]{\text{0dim.}}$ NC instanton $\xrightarrow[\text{Dirac eq.}]{\text{4dim.}}$ NC ADHM

$B_{1,2}, I, J$ $\quad V \quad$ $A_\mu = V^+ * \partial_\mu V$ $\quad \underline{\psi}$

$B'_{1,2} = B_{1,2}\,?$

$I' = I\,?, \quad J' = J\,?$

find
in terms of
the original B, I, J, V
to give the new data

$$\psi = \psi(B, I, J, V)$$

$$= V^+ * Cf$$

The answer

$$\bar{e}_\mu D_\mu * \psi = 0, \quad \int d^4x \psi^+ * \psi = 1_k$$

$$B'_{1,2} = \int d^4x\, z_{1,2} * \psi^+ * \psi = \cdots = B_{1,2}$$

$$\psi \approx \frac{I'^+, J'}{r^3} = \cdots = \frac{I^+, J}{r^3}$$

are shown

[Maeda-Sako]
[MH-Nakatsu]

F∘G=id : (inst)→(ADHM)→(inst)

NC instanton $\xrightarrow[\text{Dirac eq.}]{\text{4dim.}}$ NC ADHM $\xrightarrow[\text{Dirac eq.}]{\text{0dim.}}$ NC instanton

A_μ $\quad\psi\quad$ $B_{1,2}, I, J$ $\quad V\quad$ $A'_\mu = A_\mu$?

find V in terms of the original instanton Aμ to give the new instanton

$$V = V(A_\mu, \psi)$$

$$D^2 * V = -4\psi^+ C$$

:the answer

[Maeda-Sako] assume the existence of V.

[MH-Nakatsu] prove it

$$\nabla^+ * V = 0, \quad V^+ * V = 1_N$$

$$A'_\mu = V^+ * \partial_\mu V = \cdots = A_\mu$$

are shown (some existence proofs is also made by us)

Main result : We prove the ADHM reciprocity in the formal power series of θ-expansion for arbitrary noncommutativity (including ζ=0).

NC instanton	$\xleftrightarrow{1:1}$	NC ADHM
(i) ASD (ASDYM eq.)		(i) ASD (ADHM eq.)
(ii) C_2=k		(ii) matrix size = k, N

- This is valid only in the region that the θ-expansions converge.
- We proceed to reveal the reciprocity in operator formalism. (mostly completed [MH-Nakatsu])

G=U(N) NC ASDYM in operator formalism

- **Take coordinates as operators (in 2dim):**

$$[\hat{x},\hat{y}]=i\theta \xrightarrow{\text{complex}} [\hat{z},\hat{\bar{z}}]=2\theta \xrightarrow{\text{rescale}} [\hat{a},\hat{a}^{+}]=1$$

ann op. cre op.

acting on Fock space:

$$H=\oplus C\,\underline{|n\rangle} \quad n=0,1,2,\ldots$$

Occupation number basis

field (infinite matrix):

$$\hat{F}(\hat{z},\hat{\bar{z}})=\sum_{m,n}^{\infty}F_{mn}|m\rangle\langle n|$$

integration

$$2\pi\theta Tr_H\hat{F}(\hat{z},\hat{\bar{z}})$$

$$\hat{F}=\sum_{m_1,m_2,n_1,n_2}^{\infty}F_{m_1,m_2,n_1,n_2}|m_1,m_2\rangle\langle n_1,n_2|$$

- **NC ASDYM eq. (real rep.)**

$$\hat{F}_{01}=-\hat{F}_{23},\quad \hat{F}_{02}=-\hat{F}_{31},\quad \hat{F}_{03}=-\hat{F}_{12}$$

$$\theta^{\mu\nu}=\left[\begin{array}{cc|cc} 0 & -\theta^1 & \multicolumn{2}{c}{O} \\ \theta^1 & 0 & & \\ \hline \multicolumn{2}{c|}{O} & 0 & -\theta^2 \\ & & \theta^2 & 0 \end{array}\right] \quad \begin{array}{l} H_1 \\ \\ H_2 \end{array}$$

NC ADHM reciprocity in operator formalism

- **Strategy is almost the same as the Moyal.**
- **Set ups: the Fock representation with a regularization:** $Tr_H \hat{F}(\hat{z},\hat{\bar{z}}) = \lim_{L\to\infty} \sum_{n_1+n_2\le L} \langle n_1, n_2 | \hat{F} | n_1, n_2 \rangle$

  asymptotics: $e.g.\ \hat{A}_\mu - \hat{g}^{-1}\partial_\mu \hat{g} = O(L^{3/2}) \quad (``L \approx r^2")$

- **There are several non-trivial points:**

  $\hat{F} = \sum_{m_1,m_2,n_1,n_2}^{\infty} F_{m_1,m_2,n_1,n_2} |m_1,m_2\rangle\langle n_1,n_2|$

  Surface integration

$$\sum_{n_1+n_2\le L} (\partial^2 \hat{F})_{n_1,n_2;n_1,n_2} = \frac{8}{\zeta}\left\{(L+1)\sum_{n_1,n_2=L+1}\hat{F}_{n_1,n_2;n_1,n_2} - (L+2)\sum_{n_1,n_2=L}\hat{F}_{n_1,n_2;n_1,n_2}\right\}$$

→ **We apply this to the NC COGT formula on instanton number**

  Index theorem,...

Instanton number in NC ADHM construction

NC instanton $\xleftarrow[\text{Dirac eq.}]{\text{0dim.}}$ NC ADHM

$$\hat{\nabla}^{+}\hat{V}=0,\quad \hat{V}^{+}\hat{V}=\hat{1}_N$$

$$\hat{A}_\mu=\hat{V}^{+}\partial_\mu\hat{V}:N\times N \qquad B_{1,2}:k\times k,\quad I:k\times N,\quad J:N\times k$$

We give a natural proof of the CGOT formula in operator formalism:

$$16\pi^2 C_2 := \int d^4x\, Tr_N \hat{F}_{\mu\nu}\hat{F}_{\mu\nu} = -\int d^4x\, \partial^2\partial_\mu Tr_k \hat{f}^{-1}\partial^\mu \hat{f} = k$$

$$\hat{\nabla}=\begin{pmatrix} I^{+} & J \\ \hat{\bar{z}}_2-B_2^{+} & -(\hat{z}_1-B_1) \\ \hat{\bar{z}}_1-B_1^{+} & \hat{z}_2-B_2 \end{pmatrix}$$

(N+2k) × 2k

$$\hat{f}:=(\hat{\nabla}^{+}\hat{\nabla})^{-1}\approx r^{-2}$$

2k × 2k

We prove the NC Atiyah-Hori formula as well:

$$\int d^4x Tr_N \hat{F}_{\mu\nu}\hat{F}_{\mu\nu} = -\int d^4x Tr_{2k}\hat{\Omega}_{\mu\nu}\hat{\Omega}_{\mu\nu}$$

$$where \quad \hat{\Omega}_{\mu\nu} := \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu + [\hat{\omega}_\mu, \hat{\omega}_\nu]$$

$$\hat{\omega}_\mu := \hat{\tilde{\nabla}}^+ \partial_\mu \hat{\tilde{\nabla}}, \qquad \hat{\tilde{\nabla}} := (\hat{\nabla}^+\hat{\nabla})^{-1/2}\hat{\nabla}$$

Then:

$$C_2 := \frac{1}{16\pi^2}\int d^4x Tr_N \hat{F}_{\mu\nu}\hat{F}_{\mu\nu} = \frac{-1}{16\pi^2}\int d^4x Tr_{2k}\hat{\Omega}_{\mu\nu}\hat{\Omega}_{\mu\nu}$$

$$= \frac{1}{24\pi^2}\int_{S^3} \underline{Tr_k 1_k} \cdot Tr_2(\hat{g}^{-1}d\hat{g})^3 = k, \qquad \hat{g} := \frac{``\hat{x}^\mu e_\mu"}{r}$$

comes from the size of the ADHM data!

We note that $\hat{g}$ **is a shift operator !** $\hat{g}\hat{g}^+ = 1, \ \hat{g}^+\hat{g} = 1 - P$

This might be an origin of instanton number of NC instantons. This should be clarified.

A systematic construction of instanton

$$\hat{\nabla}^{+} = \begin{pmatrix} I & \hat{z}_2 - B_2 & -(\hat{\bar{z}}_1 - B_1^{+}) \\ J^{+} & \hat{z}_1 - B_1 & \hat{\bar{z}}_2 - B_2^{+} \end{pmatrix} =: \begin{pmatrix} \hat{\alpha}(\hat{z})^{+} \\ \beta(\hat{z}) \end{pmatrix} \qquad \hat{\alpha} \circ \hat{\beta} = 0$$

$$\hat{\nabla}^{+}\hat{V} = 0, \quad \hat{V}^{+}\hat{V} = \hat{1}_N, \quad \hat{V} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}, \qquad \hat{u} = \sum_{m_1,m_2,n_1,n_2=0}^{\infty} u(m_1,m_2,n_1,n_2) \left| m_1,m_2 \right\rangle \left\langle n_1,n_2 \right|$$

Introducing generating functions,

$$u(t_1,t_2,n_1,n_2) = \sum_{m_1,m_2=0}^{\infty} \frac{t_1^{m_1} t_2^{m_2}}{\sqrt{(m_1)!(m_2)!}} u(m_1,m_2,n_1,n_2), \quad v(t_1,t_2,n_1,n_2) = \ldots$$

we get a exact solutions without the shift operator.

(ex) G=U(1), k=1 (normalized)

$$\hat{V} = \begin{pmatrix} \hat{a}_1^{+}\hat{a}_1 + \hat{a}_2^{+}\hat{a}_2 \\ -\sqrt{\zeta}\,\hat{a}_2 \\ -\sqrt{\zeta}\,\hat{a}_1 \end{pmatrix} \sum_{\substack{(k_1,k_2)\neq(0,0) \\ (m_1,m_2)}} \left| k_1, m_1 \right\rangle \left\langle k_2, m_2 \right| \gamma_{m_1,m_2}^{k_1,k_2},$$

$$\sum_{(k_1,k_2)\neq(0,0)} (k_1+k_2)(k_1+k_2+2)\bar{\gamma}_{m_1,m_2}^{k_1,k_2} \gamma_{n_1,n_2}^{k_1,k_2} = \delta_{(m_1,m_2),(n_1,n_2)}$$

4. Conclusion and Discussion

- We discuss global solutions (instantons) of the Anti-Self-Dual Yang-Mills (ASDYM) eqs in the framework of the ADHM.

  (natural)

  NC extension → resolution of singularity

- We can construct local solutions in terms of quasideterminants in the framework of twistor

  NC extension → much easier (essential?)

- One more thing: NC ext. ←→ bkg. flux

  There is bkg. mag. field on the earth. 

Local solution of NC ASDYM

[Gilson-MH-Nimmo]

$$J_{[n]} = \begin{pmatrix} f_{[n]} - g_{[n]} b_{[n]}^{-1} e_{[n]} & -g_{[n]} b_{[n]}^{-1} \\ b_{[n]}^{-1} e_{[n]} & b_{[n]}^{-1} \end{pmatrix}$$

$$= \begin{vmatrix} \boxed{0} & -1 & 0 & \cdots & 0 & \boxed{0} \\ 1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-n} & \Delta_{-n} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-n} & \Delta_{1-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \boxed{0} & \Delta_n & \Delta_{n-1} & \cdots & \Delta_1 & \boxed{\Delta_0} \end{vmatrix}$$

Gauge fields can be reproduced from $J_{[n]}$

:compact formula in terms of quasideterminant!
(by Gelfand-Retakh)

- **Proofs are much easier than commutative one!**
- **Quasi-determinants would give more essential formulation of (NC) soliton theory including KP.**