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Inverse Mellin of ratios of Γ-s

(joint work with K. Penson)

Example:

$$W(y) := \frac{1}{\sqrt{1-y}} \cdot \left(\frac{\left(1+\sqrt{1-y}\right)^{2/3}}{y^{5/6}} + \frac{\left(1+\sqrt{1-y}\right)^{-2/3}}{y^{1/6}} \right)$$

$$h(z) := \frac{1}{\sqrt{1-z}} \left(\left(1-2z+2\sqrt{z(z-1)}\right)^{1/3} + \left(1-2z+2\sqrt{z(z-1)}\right)^{-1/3} \right)$$

imaginary for $z \in [0,1]$

but sum is real

$$h(z) = 2\sum_{n=0}^{\infty} \underbrace{\frac{(6n)!\,n!}{(3n)!\,(2n)!\,(2n)!}}_{\in \mathbb{Z}} \left(\frac{z}{108}\right)^n$$

no combinatorial interpretation is known

$$\boxed{\int_0^1 \frac{W(y)\, dy}{1-yz} = 2\pi\, h(z) \quad \forall\, z \in (0,1)}$$

General result:

Let $\alpha_j, \beta_k \in \mathbb{C}$ $\quad j,k = 1 \ldots N$

and all α_j, β_k distinct in $\mathbb{C}/\mathbb{Z}$

$$\widetilde{W}(y) := \sum_{k_0} x^{\beta_{k_0}} \frac{\prod_j \sin\left(\pi(\alpha_j - \beta_{k_0})\right)}{\prod_{k \neq k_0} \sin\left(\pi(\beta_k - \beta_{k_0})\right)} \cdot \sum_{m \geq 0} \frac{\prod_j \Gamma\left(1+m+\beta_{k_0}-\alpha_j\right)}{\prod_k \Gamma\left(1+m+\beta_{k_0}-\beta_k\right)} x^m$$

$$\Rightarrow \int_0^1 \widetilde{W}(y)\, y^s \frac{dy}{y} = \frac{\prod_k \Gamma(\beta_k + s)}{\prod_j \Gamma(\alpha_j + s)} \cdot \pi$$

<u>Proof</u>: LHS is meromorphic in s.

If $\widetilde{W}(y) \underset{y \to +0}{\sim} \sum c_\lambda y^\lambda$ $\Rightarrow$ LHS has simple poles at $s = -\lambda$ residue $= c_\lambda$

$\Rightarrow$ LHS $-$ RHS has no poles.

If $\mathrm{Re}\left(\sum \alpha_j - \sum \beta_k\right) > 1$ LHS$-$RHS uniformly bounded and $\to 0$ as $s \to +\infty$

hence $\equiv 0$.

<u>Beukers-Heckmann</u>: $\displaystyle\sum_{m \geq 0} \frac{\prod_j \Gamma(1+m+\beta_{k_0}-\alpha_j)}{\prod_k \Gamma(1+m+\beta_{k_0}-\beta_k)} x^m$ is algebraic if $\{\exp(2\pi i\, \alpha_j)\}$, $\{\exp(2\pi i\, \beta_k)\}$ interlace in $U(1)$ + $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action