Fourier Transforms of Regular Holonomic $\mathcal{D}$-Modules in Higher Dimensions

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Introduction

We study Fourier transforms of regular holonomic $\mathcal{D}$-modules in higher dimensions (this is a joint work with Kiyoshi Takeuchi). It is well-known that Fourier transforms of $\mathcal{D}$-modules preserve the holonomicity. However they do not preserve the regularity in general. In 1986 Brylinski proved that if a regular holonomic $\mathcal{D}$-module $\mathcal{M}$ is monodromic then its Fourier transform $\mathcal{M}^\wedge$ is again regular. Here we consider the more general case where the regular holonomic $\mathcal{D}$-module $\mathcal{M}$ is not necessarily monodromic.

Fourier Transforms for $\mathcal{D}$-Modules

$\mathcal{D}_X$: sheaf of algebraic differential operators on the vector space $X = \mathbb{C}^N_z$,

Y : dual space of X (i.e. $Y = \mathbb{C}^N_w$),

$W_N := \mathbb{C}[z, \partial_z]$: Weyl algebra over X,

$W_N^* := \mathbb{C}[w, \partial_w]$: Weyl algebra over Y

$(\cdot)^\wedge$: Fourier transform of $\mathcal{D}$-modules induced by the ring isomorphism:

$$W_N \xrightarrow{\sim} W_N^* \quad (z_i \to -\partial_{w_i},\ \partial_{z_i} \to w_i).$$

We obtain an equivalence of categories

$$(\cdot)^\wedge : \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_Y).$$

Condition for the Regularity

Definition 1. A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is called *monodromic* if $^\forall b$: $\mathbb{C}^*$-orbit in $X = \mathbb{C}^N$, $^\forall i \in \mathbb{Z}$, $H^i(Sol_X(\mathcal{M}))|_b$ is locally constant.

Theorem 2 (I-Takeuchi, arXiv:1807.09147)**.** *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_X$-module. Then*

$\mathcal{M}$ is monodromic $\iff$ $\mathcal{M}^\wedge$ is regular.

Remark 3.

(1) The part " $\Rightarrow$ " of Theorem 2 was proved by Brylinski. However we can reprove it by using the theory of enhanced Fourier transforms.

(2) In the course of the proof of Theorem 2 we show that the Fourier transform of a regular holonomic $\mathcal{D}_X$-module is monodromic.

Main Result 1

$i_X : X = \mathbb{C}^N \hookrightarrow \overline{X} = \mathbb{P}^N$,

$\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\overline{X}^{\mathrm{an}}})$: the triangulated category of enhanced ind-sheaves on $\overline{X}^{\mathrm{an}}$.

We define the enhanced solution complex by $Sol^{\mathrm{E}}_{\overline{X}}(\mathcal{M}) := Sol^{\mathrm{E}}_{\overline{X}^{\mathrm{an}}}\big((\mathbf{D}i_{X*}\mathcal{M})^{\mathrm{an}}\big)$ of a $\mathcal{D}_X$-module $\mathcal{M}$.

For a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we define a Zariski open subset $\Omega \subset Y$ by

$$\left\{ w \in Y \;\middle|\; \begin{array}{l} ^\exists \text{ open n.h.d } U \text{ of } w \text{ in } Y \text{ s.t.} \\ q|_{q^{-1}(U)\cap \mathrm{char}(\mathcal{M})} \text{ is an unramified covering} \end{array} \right\}.$$

For a point $w \in \Omega$, we set

$q^{-1}(w) \cap \mathrm{char}(\mathcal{M}) = \{\mu_1(w), \dots, \mu_k(w)\}$,

$\alpha_i(w) := p(\mu_i(w))$,

m_i : the multiplicity of $\mathcal{M}$ at $\mu_i(w)$.

$$\begin{array}{ccccc} X & \xleftarrow{\ p\ } & X \times Y & \xrightarrow{\ q\ } & Y \supset \Omega \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\simeq$} & & \\ \alpha_i(w) & & T^*X & \supset & \mathrm{char}(\mathcal{M}) \end{array}$$

Theorem 4 (I-Takeuchi, arXiv:1801.07444)**.** *Let U be a connected and simply connected open subset of Ω. Then we have an isomorphism in $\mathbf{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\overline{Y}^{\mathrm{an}}})$*

$$\pi^{-1}\mathbb{C}_U \otimes \left(Sol^{\mathrm{E}}_{\overline{Y}}(\mathcal{M}^\wedge)\right) \simeq$$

$$\pi^{-1}\mathbb{C}_U \otimes \bigoplus_{i=1}^{k} \text{“}\varinjlim_{a\to+\infty}\text{”}\ \mathbb{C}^{\oplus m_i}_{\{t \geq \mathrm{Re}\langle \alpha_i(w), w\rangle + a\}}.$$

In particular, $\mathcal{M}^\wedge|_\Omega$ is an integrable connection of rank $\sum_{i=1}^k m_i$.

Main Result 2

D_{reg} : the smooth part of $D := Y \setminus \Omega$,

v : a generic point of D_{reg} s.t. D is smooth hypersurface on a n.h.d. of v,

$i_M : M \hookrightarrow Y$ a normal slice of D at v s.t. $M \simeq \{u \in \mathbb{C} \mid |u| < {}^\exists\varepsilon\}$, $\{v\} = \{u = 0\}$,

$\varphi_i(u) := \langle \alpha_i(i_M(u)), i_M(u)\rangle$: holomorphic functions (Puiseux series) on $M \setminus \{v\}$,

$\mathcal{K} := \mathbf{D}i_M^*\mathcal{M}^\wedge$ for a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$.

Theorem 5 (I-Takeuchi, arXiv:1801.07444)**.** *The exponential factors in the formal decomposition of $\mathcal{K}(*\{v\})$ are the pole parts of $-\varphi_i$ $(1 \leq i \leq k)$.*
Moreover the multiplicity of the pole part of $-\varphi_i$ is equal to m_i.
In particular we have
$\mathrm{irr}(\mathcal{K}(*\{v\})) = \sum_{i=1}^k m_i \cdot \mathrm{ord}_{\{v\}}(\varphi_i)$.

$\mathcal{D}^{\mathcal{A}}$-Modules

X : complex manifold,

$\varpi : \widetilde{X} \to X$ the real blow-up of X along normal crossing divisor D,

$\mathcal{A}_{\widetilde{X}}$: sheaf of ring consisting of functions on $\widetilde{X}$ which are tempered on $\varpi^{-1}(D)$ and holomorphic on $X \setminus D$,

$\mathcal{D}^{\mathcal{A}}_{\widetilde{X}} := \mathcal{A}_{\widetilde{X}} \overset{L}{\underset{\varpi^{-1}\mathcal{O}_X}{\otimes}} \varpi^{-1}\mathcal{D}_X$,

$\mathcal{M}^{\mathcal{A}} := \mathcal{D}^{\mathcal{A}}_{\widetilde{X}} \overset{L}{\underset{\varpi^{-1}\mathcal{D}_X}{\otimes}} \varpi^{-1}\mathcal{M}$ for a $\mathcal{D}_X$-module $\mathcal{M}$.

Sectorial Irregular R-H Correspondence

Let X be a complex manifold, $D \subset X$ a normal crossing divisor and $\mathcal{M}_i$ holonomic $\mathcal{D}_X$-modules $(i = 1, 2)$.

Theorem 6 (I-Takeuchi, arXiv:1801.07444)**.** *Let W be an open subset of $\widetilde{X}$ such that $W \cap \varpi^{-1}(D) \neq \emptyset$.*
Assume $\mathcal{M}_1^{\mathcal{A}}|_W \simeq \mathcal{M}_2^{\mathcal{A}}|_W$.
Then for any sector V along D s.t.

$$\overline{\varpi^{-1}(V)} \subset W$$

there exists an isomorphism

$$\pi^{-1}\mathbb{C}_V \otimes Sol^{\mathrm{E}}_X(\mathcal{M}_1)$$
$$\simeq \pi^{-1}\mathbb{C}_V \otimes Sol^{\mathrm{E}}_X(\mathcal{M}_2).$$

We can show the converse of Theorem 6.

Theorem 7 (I-Takeuchi, arXiv:1801.07444)**.**
V : open sector in X along D,
$K_i := \pi^{-1}\mathbb{C}_V \otimes Sol^{\mathrm{E}}_X(\mathcal{M}_i)$ $(i = 1, 2)$.
Assume $K_1 \simeq K_2$.
Then for any open subset W of $\widetilde{X}$ s.t.

$$W \cap \varpi^{-1}(D) \neq \emptyset,\ \overline{W} \subset \mathrm{Int}\left(\overline{\varpi^{-1}(V)}\right)$$

there exists an isomorphism

$$\mathcal{M}_1^{\mathcal{A}}|_W \simeq \mathcal{M}_2^{\mathcal{A}}|_W.$$

Remark 8.
In the course of the proof of Theorem 6, we reconstructed $\pi^{-1}\mathbb{C}_V \otimes Sol^{\mathrm{E}}_X(\mathcal{M})$ from $\mathcal{M}^{\mathcal{A}}|_W$.
Similarly in the course of the proof of Theorem 7 we reconstructed $\mathcal{M}^{\mathcal{A}}|_W$ from $\pi^{-1}\mathbb{C}_V \otimes Sol^{\mathrm{E}}_X(\mathcal{M})$.