Simple biset functors and the double Burnside ring

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LAMFA & EPFL

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Let $B \trianglelefteq A \leq G$ and $\sigma \in \mathrm{Iso}(G, H)$. Then

$${}_G G_A\,, \quad {}_A G_G\,, \quad {}_A A/B_{A/B}\,, \quad {}_{A/B} A/B_A\,, \quad {}_H G_G$$

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Notation

For the above example,

$$ \mathrm{Ind}_A^G\,, \quad \mathrm{Res}_A^G\,, \quad \mathrm{Inf}_{A/B}^A\,, \quad \mathrm{Def}_{A/B}^A\,, \quad \mathrm{Iso}_\sigma\,. $$

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Remark

$S \in \mathcal{F}$ simple implies that $S(G)$ is a simple $kB(G, G)$-module OR zero for all finite groups G.

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- if V is simple then $J_{H,V}$ is the unique maximal subfunctor of $L_{H,V}$.
- $S_{H,V} = L_{H,V}/J_{H,V}$ is simple and all simple biset functors are obtained this way.

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Determine $\dim_k(S_{H,V}(G))$*, k is a field.*

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Then $R(-,H)$ is a subfunctor of $k\bar{B}(-,H)$.

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