

Equivariant Lefschetz formulas for smooth actions of compact groups

Ivo Dell'Ambrogio
(joint work with: Heath Emerson, Ralf Meyer)

Université de Lille 1

Colloque 2013 du GDR 2875
Topologie Algébrique et Applications
Angers
16 October 2013

References:

- I. Dell'Ambrogio, H. Emerson, R. Meyer, *An equivariant Lefschetz fixed-point formula for correspondences*, 2013 ([arXiv:1303.4777](https://arxiv.org/abs/1303.4777))
- H. Emerson, R. Meyer, a series of articles on equivariant KK-theory:
 - ▶ Math. Ann. 2006
 - ▶ J. Topol. 2009
 - ▶ Math. Ann. 2009
 - ▶ New York J. Math. 2010
 - ▶ Adv. Math. 2010 ($\times 2$)

The classical Lefschetz formula (smooth version)

- X a smooth compact manifold
- $f: X \rightarrow X$ a smooth self-map
- Assume: f has isolated non-singular fixed points p
- $\text{index}_f(p) := \deg_p(\text{id}_X - f) = \text{sign det}(\text{id}_X - d_p f)$

Theorem (Lefschetz fixed-point formula [Lefschetz, Hopf 20's])

$$\sum_p \text{index}_f(p) = \sum_i (-1)^i \text{Tr } H^i(f; \mathbb{Q}) =: \Lambda(f)$$

geometric,
local data

homological,
global data

$\Lambda(f) =$ Lefschetz number of f .

The classical Lefschetz formula, K-theory version

Consider:

- $K^*(X) = K^0(X) \oplus K^1(X)$, the topological complex K-theory of X .
- Rationally, we have the Chern isomorphism:

$$K^0(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ even}} H^n(X; \mathbb{Q}) \quad \text{and} \quad K^1(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ odd}} H^n(X; \mathbb{Q}).$$

Hence we may reformulate the Lefschetz formula in K-theoretic terms:

Theorem (Lefschetz fixed-point formula)

$$\sum_p \text{index}_f(p) = \text{Tr } K^0(f) \otimes \mathbb{Q} - \text{Tr } K^1(f) \otimes \mathbb{Q} =: \text{sTr}(K^*(f) \otimes \mathbb{Q})$$

sTr = super-trace ($\mathbb{Z}/2$ -graded trace)

What is the Lefschetz number $\Lambda(f)$, conceptually?

Conceptually: traces in symmetric tensor categories

According to [Dold-Puppe 1978], we need:

- A **symmetric monoidal category** $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1})$.
If \mathcal{C} is additive, then $\text{End}_{\mathcal{C}}(\mathbf{1})$ is a commutative ring.
- A **dualizable object** $X \in \mathcal{C}$, i.e., one admitting a *dual* object X^{\vee} and maps $\eta: \mathbf{1} \rightarrow X^{\vee} \otimes X$ and $\epsilon: X \otimes X^{\vee} \rightarrow \mathbf{1}$ yielding an adjunction:

$$\text{Hom}_{\mathcal{C}}(X \otimes -, -) \cong \text{Hom}_{\mathcal{C}}(-, X^{\vee} \otimes -).$$

- Any **endomorphism** $f: X \rightarrow X$ in \mathcal{C} .

Then we can define the **(monoidal or categorical) trace of f** to be:

$$\text{Tr}_{\otimes}(f) := \left(\begin{array}{ccccccc} \mathbf{1} & \xrightarrow{\eta} & X^{\vee} \otimes X & \xrightarrow{\text{id} \otimes f} & X^{\vee} \otimes X & & \\ & & \cong \downarrow & & \downarrow \cong & & \\ & & X \otimes X^{\vee} & \xrightarrow{f \otimes \text{id}} & X \otimes X^{\vee} & \xrightarrow{\epsilon} & \mathbf{1} \end{array} \right) \in \text{End}_{\mathcal{C}}(\mathbf{1}).$$

Conceptually: traces in symmetric tensor categories

Examples:

- 1 $\mathcal{C} = \text{Mod}(R)$ for a commutative ring R , $\otimes = \otimes_R$, $\mathbf{1} = R$, $\text{End}(\mathbf{1}) = R$
 \rightsquigarrow dualizable \equiv f. g. projective, $\text{Tr}_{\otimes}(f) = \text{usual matrix trace } \text{Tr}(f)$.
- 2 $\mathcal{C} = \text{D}(R)$, the derived category of R , $\otimes = \otimes_R^L$, $\mathbf{1} = R$, $\text{End}(\mathbf{1}) = R$.
 \rightsquigarrow dualizable \equiv perfect complexes, $\text{Tr}_{\otimes}(f) = \sum_i (-1)^i \text{Tr } H^i(f)$.
- 3 $\mathbb{Z}/2$ -graded modules $\rightsquigarrow \text{Tr}_{\otimes}(f) = \text{super trace } \text{sTr}(f)$.
- 4 stable homotopy category $\mathcal{C} = \text{SH}$, $\otimes = \wedge$, $\mathbf{1} = \Sigma^\infty S^0 \dots$
- 5 $\mathcal{C} = \text{KK}$, the **Kasparov category**, a.k.a. KK-theory (see later).

The Lefschetz number $\Lambda(f)$ can be seen as a monoidal trace in any of the above tensor categories ...

Conceptually: traces in symmetric tensor categories

General ideal: find a symmetric monoidal functor F :

$$\begin{array}{ccc} (\{\text{"compact spaces"}\}, \times, \text{pt}) & \xrightarrow{F} & (\mathcal{C}, \otimes, \mathbf{1}) \\ f: X \rightarrow X & \mapsto & F(f) \in \text{End}_{\mathcal{C}}(FX) \end{array}$$

Just make sure that:

- FX is dualizable in \mathcal{C} , for X suitably small.
- Traces can be computed effectively in \mathcal{C} .

Note: any symmetric monoidal functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ preserve traces!

- E.g., if $\text{End}(\mathbf{1})$ is torsion-free then $\text{End}(\mathbf{1}) \hookrightarrow \text{End}(\mathbf{1}) \otimes \mathbb{Q}$, so to simplify computations we can rationalize (localize): $\mathcal{C} \rightarrow \mathcal{C}_{\mathbb{Q}}$.

Question

What if X comes with a smooth action by a Lie group? Are there interesting fixed-point formulas that take the action into consideration?

GOALS

Our goal is to generalize the Lefschetz formula in the context of KK-theory.

The generalization is in two different directions:

- 1 by allowing a compact Lie group G to act on X .
For G -equivariant f , we will have $\text{Tr}_{\otimes}(f) \in R(G)$ = the complex representation ring (character ring) of G .
- 2 by allowing not only maps $f: X \rightarrow X$, but general smooth G -equivariant correspondences $\varphi = [X \xleftarrow{f} (M, \xi) \xrightarrow{g} X]$.

Prototype of the generalized formula:

For $\varphi: X \rightarrow X$ a correspondence (e.g. $\varphi = [f]$ from a map f) we compute $\text{Tr}_{\otimes}(\varphi)$ in the G -equivariant Kasparov category KK^G in two ways:

$$\text{Tr}_{\otimes}(f) = \boxed{\text{geometric \& local}} = \boxed{\text{homological \& global}} \in R(G).$$

The equivariant Kasparov category

Let G be a compact Lie group.

The G -equivariant Kasparov category, KK^G , has the following properties:

- 1 Its objects are the separable complex G - C^* -algebras.
For instance: X (locally) compact G -space $\Rightarrow C_0(X) \in KK^G$.
- 2 Hom-sets: $KK^G(A, B)$, Kasparov's G -equivariant KK -theory groups.
E.g: $KK^G(\mathbb{C}, C_0(X)) \cong K_G^0(X)$ is equivariant K -theory.
- 3 KK^G is additive, indeed it is a triangulated category ($\Sigma^2 \cong \text{id}$).
- 4 KK^G is a tensor category: $A \otimes B = A \hat{\otimes}_{\min} B$, $\mathbf{1} = \mathbb{C}$.

We have a symmetric monoidal functor:

$$(\{(\text{locally} \text{ compact } G\text{-spaces} \}, \times, \text{pt})^{\text{op}} \xrightarrow{C_0(-)} KK^G$$

If X is a smooth compact G -manifold then $C_0(X) = C(X)$ is dualizable (can construct a dual geometrically).

- 5 $\text{End}_{KK^G}(\mathbf{1}) \cong R(G)$, the complex representation ring of G .

The Geometric side of the equation (sketch)

[Emerson-Meyer 2006-10] **Category of smooth correspondences, \widehat{KK}^G** :

- Objects: smooth G -manifolds.
- Morphisms: smooth G -equivariant correspondences

$$\widehat{KK}^G(X, Y) \ni \varphi = [X \xleftarrow{f} M \xrightarrow{g} Y, \xi]$$

Roughly speaking:

- ▶ $f: M \rightarrow X$ a smooth G -map
- ▶ $g: M \rightarrow Y$ a G -map with a nice factorization (+ Spin^c structures)
- ▶ $\xi \in RK_{G,X}^*(M)$ a K-theory class with X -compact supports
- Composition: intersection product (pull-back).
- Have monoidal functors:

$$\begin{array}{ccc} & C_0(-) & \\ & \curvearrowright & \\ \{G\text{-manifolds}\}^{\text{op}} & \xrightarrow{f \mapsto [f, g=\text{id}, \xi=1]} & \widehat{KK}^G \longrightarrow KK^G \end{array}$$

- $\widehat{KK}^G(X, Y) \xrightarrow{\cong} KK^G(C_0(X), C_0(Y))$, if X has finite orbit type, e.g. if X is compact.

The Geometric side of the equation (sketch)

X a compact smooth G -manifold.

$\varphi = [X \xleftarrow{f} M \xrightarrow{g} X, \xi]$ a correspondence $X \rightarrow X$.

- **Coincidence space** $Q := \{m \in M \mid f(m) = g(m)\}$
- Assume that f, g *intersect smoothly*:
 $Q \subseteq M$ smooth sub-mf, and $\forall v \in TM, df(v) = dg(v) \Rightarrow v \in TQ$.
- **Excess bundle** $\eta := \text{Coker}(df - dg) \in K_G^0(Q)$, a G -bundle on Q .

Theorem 1 (Emerson-Meyer 2013)

With the above hypotheses and notations, we have:

$$\text{Tr}_{\otimes}(\varphi) = \left(\begin{array}{l} \text{the } G\text{-index of the Dirac operator on } Q \\ \text{twisted by the bundle } \xi|_Q \otimes \eta \in K_G^*(Q) \end{array} \right) \in R(G)$$

For $G = 1$ and $\varphi = [f, \text{id}, 1]$ where f has isolated non-singular fixed-points, this reduce to $\sum_{p \in Q} \text{index}_f(x)$.

The Homological side: preliminaries

$S^{-1}R(G)$: the **total ring of fractions**, $S := \{\text{nonzero divisors of } R(G)\}$.
Obtain an inclusion $R(G) \hookrightarrow S^{-1}R(G)$ into a product of fin. many fields:

$$S^{-1}R(G) \cong \prod_{\mathfrak{p}_i \text{ minimal prime}} \underbrace{\text{Frac}(R(G)/\mathfrak{p}_i)}_{=: F_i, \text{ a field.}}$$

Have a bijection: $\{\mathfrak{p}_i\} \leftrightarrow \{\text{cyclic subgroups of } G/G^0\}$
In particular, $S^{-1}R(G)$ is a field iff G is connected.

Since $R(G) = \text{End}(\mathbf{1})$ acts on KK^G , we can localize the whole category KK^G at S to obtain a (simpler!) tensor-triangulated category:

$$KK^G \xrightarrow{\text{symm. mon.}} S^{-1}KK^G$$

By injectivity, it will suffice to compute traces on the right-hand side.

The Homological side: abstract result

For any $A \in KK^G$, there is a decomposition of $S^{-1}R(G)$ -modules:

$$S^{-1}K_*^G(A) \stackrel{\text{def.}}{=} S^{-1}KK_*^G(\mathbb{C}, A) \cong \bigoplus_i K_{*,i}^G(A)$$

where $K_{*,i}^G(A)$ is a $\mathbb{Z}/2$ -graded vector space over F_i .

Theorem 2 ([D.-E.-M. 2013])

Assume that $A \in KK^G$ belongs to the thick subcategory generated by \mathbb{C} , $\text{Thick}(\mathbb{C})$. Then A is dualizable and for any $\varphi \in KK^G(A, A)$ we have

$$\text{Tr}_{\otimes}(\varphi) = \left(\text{sTr } K_{*,i}^G(\varphi) \right)_i \in \prod_i F_i = S^{-1}R(G)$$

where $K_{,i}^G(\varphi): K_{*,i}^G(A) \rightarrow K_{*,i}^G(A)$ is the F_i -linear map induced by φ .*

The Homological side: Hodgkin Lie groups

Assume G is a **Hodgkin** Lie group: compact, connected, $(\pi_1 G)_{\text{tors}} = 0$.
Examples: $G = \mathbb{T} = U(1)$ the circle group or $G = \mathbb{T}^n$ a torus.

Theorem 3 ([DEM 2013])

Let $A \in KK^G$ for G Hodgkin. Then $A \in \text{Thick}(\mathbb{C})$ iff:

(1) A is dualizable, and (2) $\text{Res}_1^G(A)$ belongs to the Bootstrap category.

(2) is satisfied e.g. if $A = C_0(X)$ is commutative.

(1) is satisfied if moreover X is a smooth G -manifold (\exists explicit dual).

Corollary 4 ([DEM 2013])

Let G be a Hodgkin Lie group and X a smooth compact G -manifold.

For any correspondence $\varphi \in \widehat{KK}^G(X, X)$ we have

$$\text{Tr}_{\otimes}(\varphi) = \text{sTr}(S^{-1}K_*^G(\varphi)) \in S^{-1}R(G) = \text{Frac } R(G).$$

The Homological side: other groups?

This fails badly for non-connected groups!

Example

Let $G \neq 1$ be any finite group, e.g. $G = \mathbb{Z}/2$. Then $A = C(G) \in KK^G$ is *not* contained in $\text{Thick}(\mathbb{C})$ (although $C(G)$ is certainly dualizable). Worse: the given trace formula is wrong for $\varphi \in KK^G(C(G), C(G)) \cong \mathbb{Z}[G]$!

For $\varphi \equiv g \in G$, we have $\text{Tr}_{\otimes}(\varphi) = \begin{cases} [\mathbb{C}G] \in R(G) & \text{if } g = 1 \\ 0 & \text{if } g \neq 1. \end{cases}$

But $K_*^G(C(G)) = K_*^*(G) \cong \mathbb{Z}$ and $K_*^G(\varphi) = \text{id}$ for all g .

Solution: consider $K_*^H(\varphi)$ for sufficiently many subgroups $H \leq G$: this solves both issues at the same time! In fact, it suffices to consider a certain subgroup H_i for each minimal prime ideal \mathfrak{p}_i of $R(G)$.

The Homological side: general groups

Now let G be **any** compact Lie group.

Recall from [Segal 1968]:

$$\begin{aligned} \{\text{minimal primes } \mathfrak{p} \subset R(G)\} &\xrightarrow{\sim} \{\text{cyclic subgps. } C \leq G/G^0\}/\text{conj.} \\ &\xrightarrow{\sim} \{\text{Cartan subgps. } H \leq G\}/\text{conj.} \end{aligned}$$

H Cartan: topologically cyclic ($H \cong \mathbb{T}^r \times \mathbb{Z}/k$) and $|N_G(H)/H| < \infty$.

$$S^{-1}R(G) \cong \prod_{H \text{ Cartan} / \text{conj.}} \underbrace{\text{Frac}(R(G)/\mathfrak{p}_H)}_{=: F_H}$$

If $A \in KK^G$, then:

- $K_*^H(A) \stackrel{\text{def.}}{=} KK_*^H(\mathbb{C}, \text{Res}_H^G A)$ is an $R(G)$ -module via $R(G) \xrightarrow{\text{res}} R(H)$.
- $K_*^H(A) \otimes_{R(G)} F_H$ is an F_H -vector space.

The Homological side: general groups

Definition (The G -equivariant Bootstrap class)

$$\mathcal{B}^G := \text{Loc}\{\text{Ind}_H^G(M_n(\mathbb{C})) : H \leq G \text{ closed}, H \curvearrowright M_n(\mathbb{C}) \text{ any action}\} \subseteq KK^G$$

Note that \mathcal{B}^G contains $C(X)$ for all compact G -manifolds, but also many non-commutative G - C^* -algebras.

Theorem 5 ([DEM 2013])

Let $A \in KK^G$ be a dualizable algebra in \mathcal{B}^G . Then for any $\varphi \in KK^G(A, A)$ we have

$$\text{Tr}_{\otimes}(\varphi) = \left(\text{sTr} (K_*^H(\varphi) \otimes_{R(G)} F_H) \right)_H \in \prod_{H \text{ Cartan/conj.}} F_H = S^{-1}R(G)$$

where $K_*^H(\varphi) \otimes_{R(G)} F_H$ is the map induced by φ on $K_*^H(A) \otimes_{R(G)} F_H$.