Coherent presentations of Artin groups

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Motivation

▶ A **Coxeter system** $(\mathbf{W}, S)$ is a data made of a group $\mathbf{W}$ with a presentation by a (finite) set S of involutions, $s^2 = 1$, satisfying **braid relations**

$$tstst\ldots = ststs\ldots$$

▶ Forgetting the involutive character of generators, one gets the **Artin's presentation** of the **Artin group**

$$\mathbf{B}(\mathbf{W}) = \langle\ S \mid tstst\ldots = ststs\ldots\rangle$$

Objective.

- Push further Artin's presentation and study the **relations among the braid relations**. (Brieskorn-Saito, 1972, Deligne, 1972, Tits, 1981).

- We introduce a rewriting method to compute generators of relations among relations.

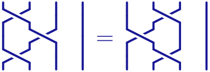
Motivation

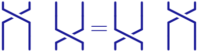
- Set $\mathbf{W} = \mathbf{S}_4$ the group of permutations of $\{1, 2, 3, 4\}$, with $S = \{r, s, t\}$ where

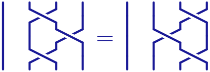
$r =$  $s =$  $t =$ 

- The associated Artin group is the group of braids on 4 strands

$$\mathbf{B}(\mathbf{S}_4) = \langle\, r, s, t \mid rsr = srs, \;\; rt = tr, \;\; tst = sts \,\rangle$$







- The relations among the braid relations on 4 strands are generated by the **Zamolodchikov relation** (Deligne, 1997).

$$stsrst = tstrst = tsrtst = tsrsts = trsrts = rtstrs = rstsrs = rstrsr = rsrtsr = srstsr = srtstr = strsrt = stsrst \qquad Z_{r,s,t}$$

Plan

I. Coherent presentations of categories
- Polygraphs as higher-dimensional rewriting systems
- Coherent presentations as cofibrant approximation

II. Homotopical completion-reduction procedure
- Tietze transformations
- Rewriting properties of 2-polygraphs
- The homotopical completion-procedure

III. Applications to Artin groups
- Garside's coherent presentation
- Artin's coherent presentation

References

- S. Gaussent, Y. Guiraud, P. M., Coherent presentations of Artin groups, ArXiv preprint, 2013.
- Y. Guiraud, P.M., Higher-dimensional normalisation strategies for acyclicity, Adv. Math.,2012.

Part I. Coherent presentations of categories

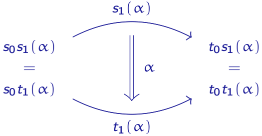
Polygraphs

- A **1-polygraph** is an oriented graph (Σ_0, Σ_1)

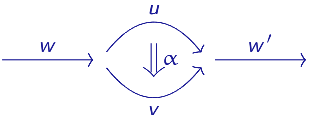
$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1$$

- A **2-polygraph** is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ where
  - (Σ_0, Σ_1) is a 1-polygraph,
  - Σ_2 is a globular extension of the free category Σ_1^*.

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2$$



- A **rewriting step** is a 2-cell of the free 2-category Σ_2^* over Σ with shape

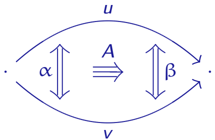


where $\alpha : u \Rightarrow v$ is a 2-cell of Σ_2 and w, w' are 1-cells of Σ_1^*.

Polygraphs

- A **$(3,1)$-polygraph** is a pair $\Sigma = (\Sigma_2, \Sigma_3)$ made of
  - a 2-polygraph Σ_2,
  - a globular extension Σ_3 of the free $(2,1)$-category $\Sigma_2^\top$.

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2^\top \underset{t_2}{\overset{s_2}{\leftleftarrows}} \Sigma_3$$



Let $\mathbf{C}$ be a category.

- A **presentation** of $\mathbf{C}$ is a 2-polygraph Σ such that

$$\mathbf{C} \simeq \Sigma_1^*/\Sigma_2$$

- An **extended presentation** of $\mathbf{C}$ is a $(3,1)$-polygraph Σ such that

$$\mathbf{C} \simeq \Sigma_1^*/\Sigma_2$$

Coherent presentations of categories

▶ A **coherent presentation** of $\mathbf{C}$ is an extended presentation Σ of $\mathbf{C}$ such that the cellular extension Σ_3 is a **homotopy basis**.

In other words

- the quotient $(2,1)$-category $\Sigma_2^{\top}/\Sigma_3$ is aspherical,

- the congruence generated by Σ_3 on the $(2,1)$-category $\Sigma_2^{\top}$ contains every pair of parallel 2-cells.

Example. The full coherent presentation contains all the 3-cells.

Theorem. [Gaussent-Guiraud-M., 2013]

Let Σ be an extended presentation of a category $\mathbf{C}$. Consider the Lack's model structure for 2-categories.
The following assertions are equivalent:

i) *The $(3,1)$-polygraph Σ is a coherent presentation of $\mathbf{C}$.*

ii) *The $(2,1)$-category presented by Σ is a cofibrant 2-category weakly equivalent to $\mathbf{C}$, that is a **cofibrant approximation** of $\mathbf{C}$.*

Examples

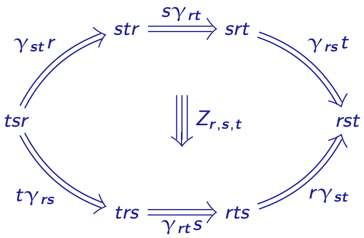
- Free monoid : no relation, an empty homotopy basis.
- Free commutative monoid:

$$\mathbb{N}^3 = \langle\, r, s, t \mid sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\ ts \overset{\gamma_{st}}{\Longrightarrow} st,\ tr \overset{\gamma_{rt}}{\Longrightarrow} rt \mid \begin{array}{c}\text{all the}\\ \text{3-cells}\end{array} \,\rangle$$

- A homotopy basis can be made with only one 3-cell

$$\mathbb{N}^3 = \langle\, r, s, t \mid sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\ ts \overset{\gamma_{st}}{\Longrightarrow} st,\ tr \overset{\gamma_{rt}}{\Longrightarrow} rt \mid Z_{r,s,t} \,\rangle$$

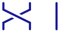
is a coherent presentation, where $Z_{r,s,t}$ is the **permutaedron**



Examples

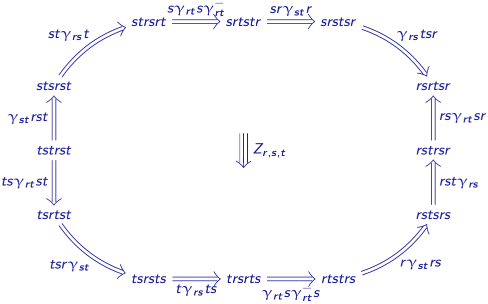
▶ **Artin's coherent presentation** of the monoid $\mathbf{B}_3^+$

$$\mathrm{Art}_3(\mathbf{S}_3) = \langle\, s, t \mid tst \overset{\gamma_{st}}{\Longrightarrow} sts \mid \emptyset \,\rangle$$

where $s =$  and $t =$ 

▶ **Artin's coherent presentation** of the monoid $\mathbf{B}_4^+$

$$\mathrm{Art}_3(\mathbf{S}_4) = \langle\, r, s, t \mid rsr \overset{\gamma_{sr}}{\Longrightarrow} srs,\ \ rt \overset{\gamma_{tr}}{\Longrightarrow} tr,\ \ tst \overset{\gamma_{st}}{\Longrightarrow} sts \mid Z_{r,s,t} \,\rangle$$



Coherent presentations

Problems.

1. How to transform a coherent presentation ?
2. How to compute a coherent presentation ?

Part II. Homotopical completion-reduction procedure

Tietze transformations

▶ Two $(3,1)$-polygraphs Σ and Υ are **Tietze-equivalent** if there is an equivalence of 2-categories

$$F : \Sigma_2^\top / \Sigma_3 \longrightarrow \Upsilon_2^\top / \Upsilon_3$$

inducing an isomorphism on presented categories.

▶ In particular, two coherent presentations of the same category are Tietze-equivalent.

▶ An **elementary Tietze transformation** of a $(3,1)$-polygraph Σ is a 3-functor with source $\Sigma^\top$ that belongs to one of the following three pairs of dual operations:

▶ **add a generator**: for $u \in \Sigma_1^*$, add a generating 1-cell x and add a generating 2-cell

$$u \overset{\delta}{\Longrightarrow} x$$

▶ **remove a generator**: for a generating 2-cell α in Σ_2 with $x \in \Sigma_1$, remove x and α

$$u \overset{\cancel{\alpha}}{\Longrightarrow} x\cancel{}$$

Tietze transformations

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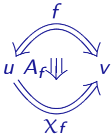
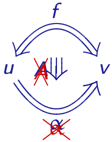
- An **elementary Tietze transformation** of a $(3,1)$-polygraph Σ is a 3-functor with source $\Sigma^\top$ that belongs to one of the following three pairs of dual operations:

- **add a relation**: for a 2-cell $f \in \Sigma_2^\top$, add a generating 2-cell χ_f add a generating 3-cell A_f



- **remove a relation**: for a 3-cell A where $\alpha \in \Sigma_2$, remove α and A



Tietze transformations

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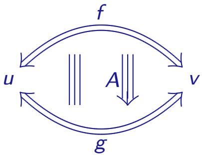
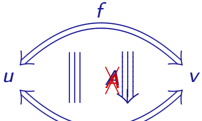
▶ In particular, two coherent presentations of the same category are Tietze-equivalent.

▶ An **elementary Tietze transformation** of a $(3,1)$-polygraph Σ is a 3-functor with source $\Sigma^{\top}$ that belongs to one of the following three pairs of dual operations:

▶ **add a 3-cell**: for equals 2-cells $f \equiv g$, add a generating 3-cell $f \overset{A}{\Rrightarrow} g$



▶ **remove a 3-cell**: for a generating 3-cell $f \overset{A}{\Rrightarrow} g$ with $f \equiv g$, remove A



Tietze transformations

Theorem. [Gaussent-Guiraud-M., 2013]

Two (finite) $(3,1)$-polygraphs Σ and Υ are Tietze equivalent if, and only if, there exists a (finite) Tietze transformation

$$\mathcal{T} : \Sigma^{\top} \to \Upsilon^{\top}$$

Consequence.

If Σ is a coherent presentation of a category $\mathbf{C}$ and if there exists a Tietze transformation

$$\mathcal{T} : \Sigma^{\top} \to \Upsilon^{\top}$$

then Υ is a coherent presentation of $\mathbf{C}$.

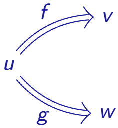
Rewriting properties of 2-polygraphs

Let $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ be a 2-polygraph.

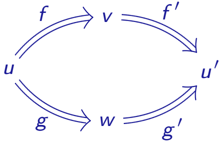
- Σ **terminates** if it does not generate any infinite reduction sequence

$$u_1 \Longrightarrow u_2 \Longrightarrow \cdots \Longrightarrow u_n \Longrightarrow \cdots$$

- A **branching** of Σ is a pair (f, g) of 2-cells of Σ_2^* with a common source



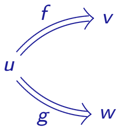
- Σ is **confluent** if all of its branchings are confluent:



- Σ is **convergent** if it terminates and it is confluent.

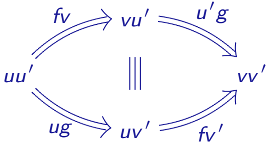
Rewriting properties of 2-polygraphs

- A branching



is **local** if f and g are rewriting steps.

- Local branchings are classified as follows:
  - **aspherical** branchings have shape (f, f),
  - **Peiffer** branchings have shape



$$fv \star_1 u'g = ug \star_1 fv'$$

  - **overlap** branchings are all the other cases.

- A **critical branching** is a minimal (for inclusion of source) overlap branching.

Theorem. [Newman's diammond lemma, 1942]

For terminating 2-polygraphs, local confluence and confluence are equivalent properties.

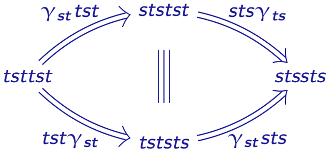
Rewriting properties of 2-polygraphs

Example.

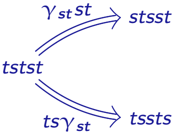
Consider the 2-polygraph

$$\mathrm{Art}_2(\mathbf{S}_3) = \langle\, s, t \mid tst \overset{\gamma_{st}}{\Longrightarrow} sts \,\rangle$$

- A Peiffer branching:



- A critical branching:



Homotopical completion procedure

▶ The **homotopical completion** procedure is defined as the combinaison of the

- **Knuth-Bendix procedure** computing a convergent presentation from a terminating presentation (Knuth-Bendix, 1970).

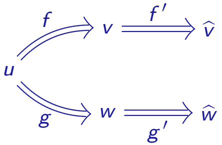
- **Squier procedure**, computing a coherent presentation from a convergent presentation (Squier, 1994).

Homotopical completion procedure

Let Σ be a terminating 2-polygraph (with a total termination order).

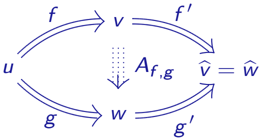
▶ The **homotopical completion** of Σ is the $(3,1)$-polygraph $\mathcal{S}(\Sigma)$ obtained from Σ by successive application of following Tietze transformations

- for every critical pair

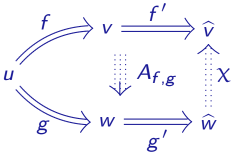


compute f' and g' reducing to some normal forms.

- if $\widehat{v} = \widehat{w}$, add a 3-cell $A_{f,g}$



- if $\widehat{v} < \widehat{w}$, add the 2-cell χ and the 3-cell $A_{f,g}$



Homotopical completion procedure

- Potential adjunction of additional 2-cells χ can create new critical branchings,
  - whose confluence must also be examined,
  - possibly generating the adjunction of additional 2-cells and 3-cells
  - ...

- This defines an increasing sequence of $(3,1)$-polygraphs

$$\langle \Sigma \mid \emptyset \rangle \;=\; \Sigma^{0} \;\subseteq\; \Sigma^{1} \;\subseteq\; \cdots \;\subseteq\; \Sigma^{n} \;\subseteq\; \Sigma^{n+1} \;\subseteq\; \cdots$$

- The **homotopical completion** of Σ is the $(3,1)$-polygraph

$$\mathcal{S}(\Sigma) = \bigcup_{n \geqslant 0} \Sigma^{n}.$$

Theorem. [Gaussent-Guiraud-M., 2013]

For every terminating presentation Σ of a category $\mathbf{C}$, the homotopical completion $\mathcal{S}(\Sigma)$ of Σ is a coherent convergent presentation of $\mathbf{C}$.

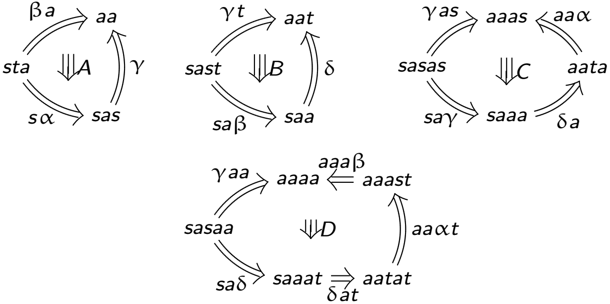
Homotopical completion procedure

Example. The **Kapur-Narendran presentation** of $\mathbf{B}_3^+$, obtained from Artin's presentation by coherent adjunction of the Coxeter element st

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \rangle$$

The deglex order generated by $t > s > a$ proves the termination of Σ_2^{KN}.

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A,\ B,\ C,\ D \,\rangle$$



However. The extended presentation $\mathcal{S}(\Sigma_2^{\mathrm{KN}})$ obtained is bigger than necessary.

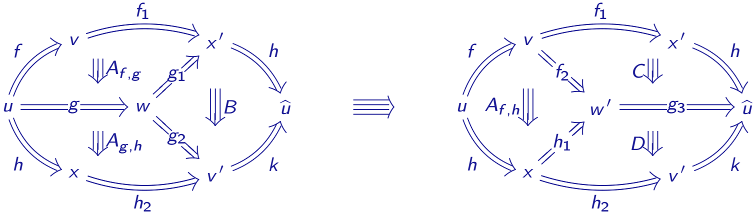
Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

Step 2. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,



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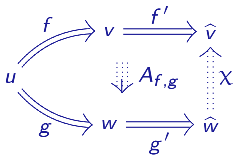
- 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,

- the 3-cells adjoined with a 2-cell by homotopical completion to reach confluence,

- some collapsible 2-cells or 3-cells already present in the initial presentation Σ.



Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

Step 2. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,
- the 3-cells adjoined with a 2-cell by homotopical completion to reach confluence,
- some collapsible 2-cells or 3-cells already present in the initial presentation Σ.

The **homotopical completion-reduction** of terminating 2-polygraph Σ is the $(3,1)$-polygraph

$$\mathcal{R}(\Sigma) = \pi_{\Gamma}(\mathcal{S}(\Sigma))$$

Theorem. [Gaussent-Guiraud-M., 2013]

For every terminating presentation Σ of a category $\mathbf{C}$, the homotopical completion-reduction $\mathcal{R}(\Sigma)$ of Σ is a coherent convergent presentation of $\mathbf{C}$.

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

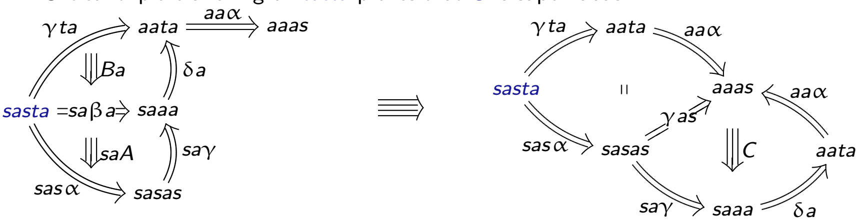
$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, \cancel{C}, D \,\rangle$$

- There are four critical triple branchings, overlapping on

$$sasta,\ sasast,\ sasasas,\ sasasaa.$$

  - Critical triple branching on $sasta$ proves that C is superfluous:



$$C = sas\alpha^{-1} \star_1 (Ba \star_1 aa\alpha) \star_2 (saA \star_1 \delta a \star_1 aa\alpha)$$

The homotopical completion-reduction procedure

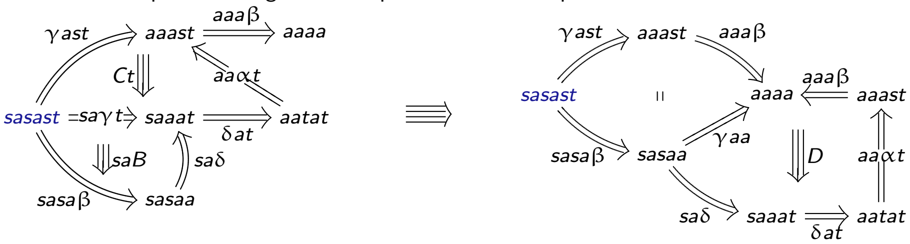
Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as, st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, \cancel{C}, \cancel{D} \,\rangle$$

- Critical triple branching on $sasast$ proves that D is superfluous:



$$D = sasa\beta^{-1} \star_1 \big((Ct \star_1 aaa\beta) \star_2 (saB \star_1 \delta at \star_1 aa\alpha t \star_1 aaa\beta) \big)$$

The homotopical completion-reduction procedure

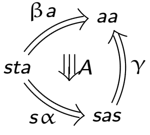
Example.

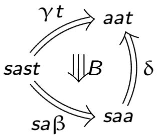
$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a\ \rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\ s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D\ \rangle$$

$$\langle\ s, t, a \mid\ ta \overset{\alpha}{\Longrightarrow} as\ , st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\not\Longrightarrow} aa,\ , saa \overset{\delta}{\not\Longrightarrow} aat \mid \not A, \not B, \not C, \not D \rangle$$

- The 3-cells A and B correspond to the adjunction of the rules γ and δ during the completion





The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, \cancel{a} \mid tst \overset{\alpha}{\Longrightarrow} sts, \cancel{st \overset{\beta}{\Longrightarrow} a}, \cancel{sas \overset{\gamma}{\Longrightarrow} aa}, \cancel{saa \overset{\delta}{\Longrightarrow} aat} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D} \,\rangle$$

- The generator a is superflous.

$$\mathcal{R}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t \mid tst \overset{\alpha}{\Longrightarrow} sts \mid \emptyset \,\rangle$$

$$= \mathrm{Art}_3(\mathbf{S}_3)$$

Part III. Applications to Artin groups

Garside's presentation

- Let $\mathbf{W}$ be a Coxeter group

$$\mathbf{W} = \langle\, S \mid s^2 = 1,\ \ \langle ts \rangle^{m_{st}} = \langle st \rangle^{m_{st}}, \quad \text{with } m_{st} \neq \infty \text{ and } s, t \in S \,\rangle$$

where $\langle ts \rangle^{m_{st}}$ stands for the word $tsts\ldots$ with m_{st} letters.

- **Artin's presentation** of the Artin monoid $\mathbf{B}^+(\mathbf{W})$:

$$\mathrm{Art}_2(\mathbf{W}) = \langle\, S \mid \langle ts \rangle^{m_{st}} = \langle st \rangle^{m_{st}}, \quad \text{with } m_{st} \neq \infty \text{ and } s, t \in S \,\rangle$$

- **Garside's presentation** of $\mathbf{B}^+(\mathbf{W})$

$$\mathrm{Gar}_2(\mathbf{W}) = \langle\, \mathbf{W} \setminus \{1\} \mid u|v \overset{\alpha_{u,v}}{\Longrightarrow} uv, \text{ whenever } u \frown v \,\rangle$$

where

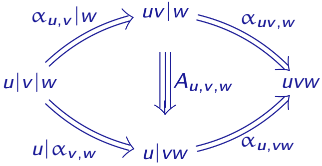
uv is the product in $\mathbf{W}$,

$u|v$ is the product in the free monoid over $\mathbf{W}$.

- Notations :
  - $u \frown v$ whenever $l(uv) = l(u) + l(v)$.
  - $u \overset{\times}{} v$ whenever $l(uv) < l(u) + l(v)$.

Garside's coherent presentation

▶ The **Garside's coherent presentation** of $\mathbf{B}^+(\mathbf{W})$ is the extended presentation $\mathrm{Gar}_3(\mathbf{W})$ obtained from $\mathrm{Gar}_2(\mathbf{W})$ by adjunction of one 3-cell



for every u, v, w in $\mathbf{W} \setminus \{1\}$ with $u \; v \; w$.

Theorem. [Gaussent-Guiraud-M., 2013]

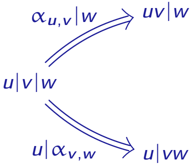
For every Coxeter group $\mathbf{W}$*, the Artin monoid* $\mathbf{B}^+(\mathbf{W})$ *admits* $\mathrm{Gar}_3(\mathbf{W})$ *as a coherent presentation.*

Proof. By homotopical completion-reduction of the 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$.

Garside's coherent presentation

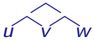
Step 1. We compute the coherent convergent presentation $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$

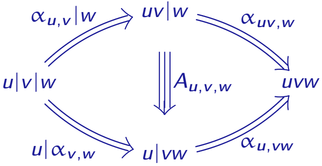
- The 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$ has one critical branching for every u, v, w in $\mathbf{W} \setminus \{1\}$ when



- There are two possibilities.

if

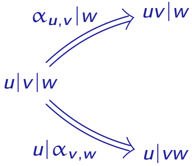




Garside's coherent presentation

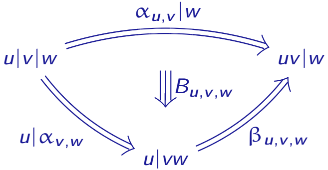
Step 1. We compute the coherent convergent presentation $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$

- The 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$ has one critical branching for every u, v, w in $\mathbf{W} \setminus \{1\}$ when

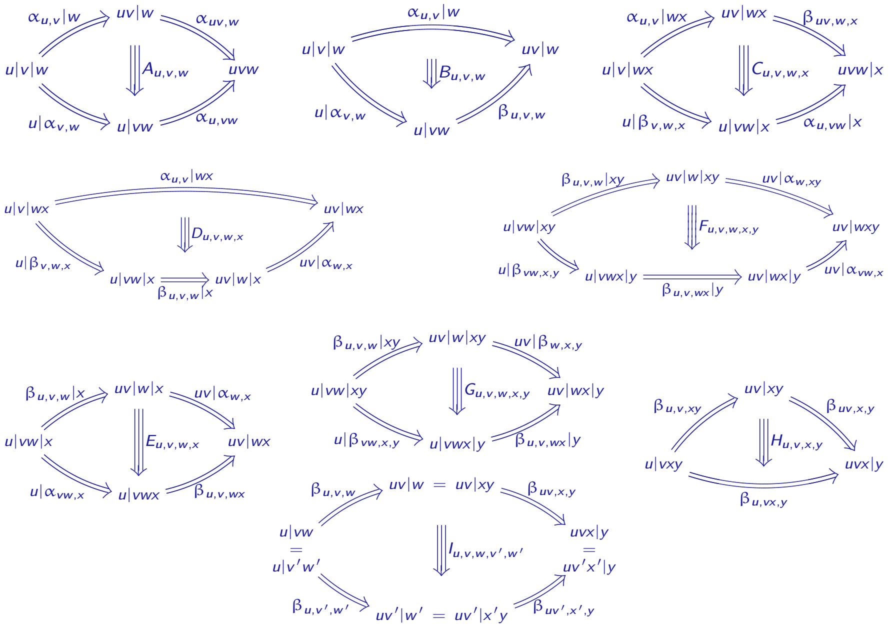


- There are two possibilities.

otherwise



Garside's coherent presentation



Garside's coherent presentation

Proposition.

For every Coxeter group $\mathbf{W}$, the Artin monoid $\mathbf{B}^+(\mathbf{W})$ admits, as a coherent convergent presentation the $(3,1)$-polygrap $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$ where

- the 1-cells are the elements of $\mathbf{W} \setminus \{1\}$,

- there is a 2-cell $u|v \overset{\alpha_{u,v}}{\Longrightarrow} uv$ for every u, v in $\mathbf{W} \setminus \{1\}$ with $u \frown v$,

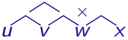
- the 2-cells $u|vw \overset{\beta_{u,v,w}}{\Longrightarrow} uv|w,$ for every u, v, w in $\mathbf{W} \setminus \{1\}$ with $u \frown \overset{\times}{v} \frown w$,

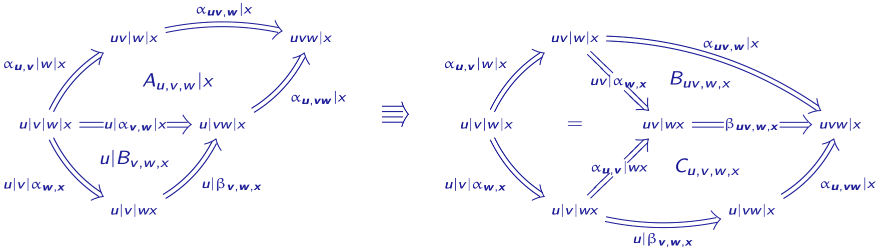
- the nine families of 3-cells A, B, C, D, E, F, G, H, I.

Garside's coherent presentation

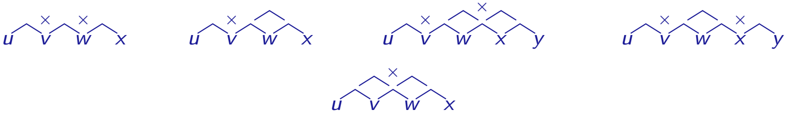
Step 2. Homotopical reduction of $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$.

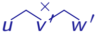
- We consider some triple critical branchings of $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$

In the case 



and similar 3-spheres for the following cases



 and  with $vw = v'w'$

Artin's coherent presentation

Theorem. [Gaussent-Guiraud-M., 2013]

For every Coxeter group $\mathbf{W}$, *the Artin monoid* $\mathbf{B}^{+}(\mathbf{W})$ *admits the coherent presentation* $\mathrm{Art}_3(\mathbf{W})$ *made of*

- *Artin's presentation* $\mathrm{Art}_2(\mathbf{W}) = \langle\, S \mid \langle ts \rangle^{m_{st}} = \langle st \rangle^{m_{st}},\, \rangle$

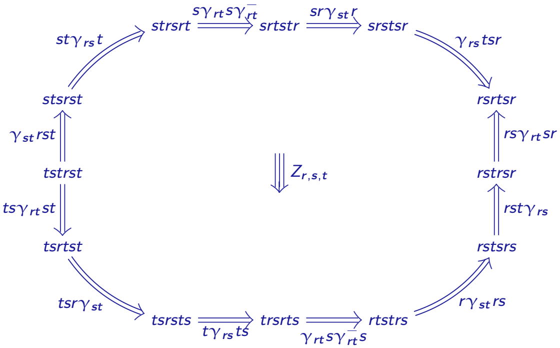
- *one* 3*-cell* $Z_{r,s,t}$ *for every elements* $t > s > r$ *of* S *such that the subgroup* $\mathbf{W}_{\{r,s,t\}}$ *is finite.*

Theorem. [Gaussent-Guiraud-M., 2013]

For every Coxeter group $\mathbf{W}$, *the Artin group* $\mathbf{B}(\mathbf{W})$ *admits* $\mathrm{Gar}_3(\mathbf{W})$ *and* $\mathrm{Art}_3(\mathbf{W})$ *as coherent presentation.*

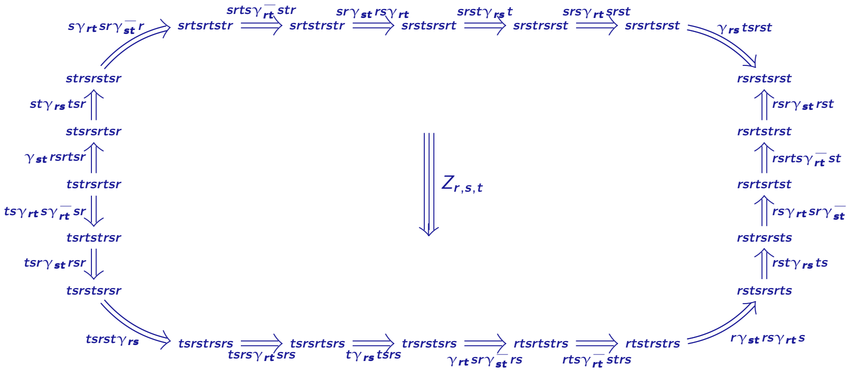
Artin's coherent presentation

The 3-cells $Z_{r,s,t}$ for Coxeter types A_3



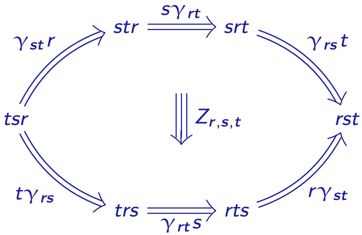
Artin's coherent presentation

The 3-cells $Z_{r,s,t}$ for Coxeter types B_3



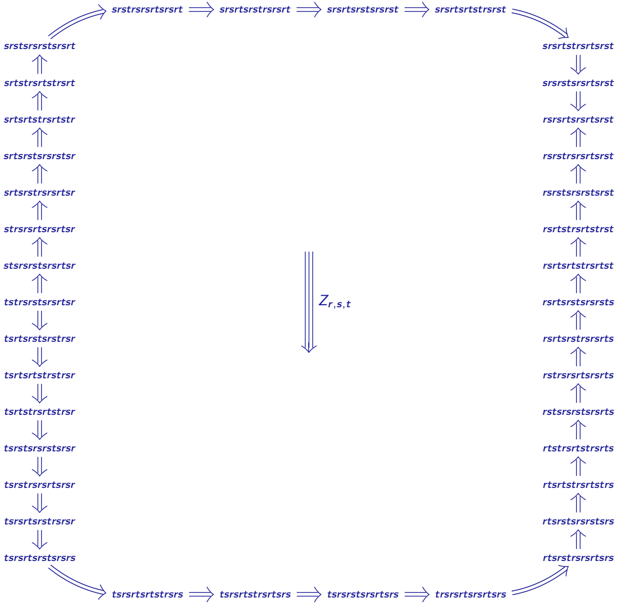
Artin's coherent presentation

The 3-cells $Z_{r,s,t}$ for Coxeter types $A_1 \times A_1 \times A_1$



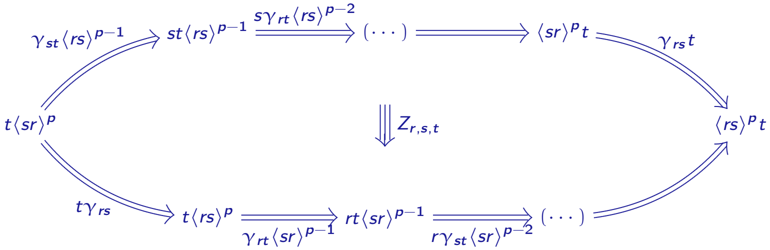
Artin's coherent presentation

The 3-cells $Z_{r,s,t}$ for Coxeter type H_3



Artin's coherent presentation

The 3-cells $Z_{r,s,t}$ for Coxeter type $I_2(p) \times A_1$, $p \geqslant 3$



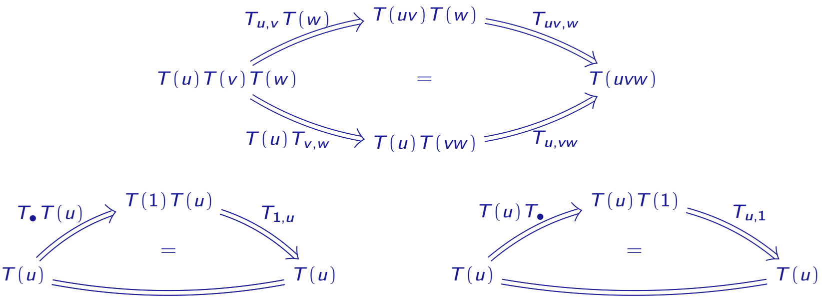
Coherent presentations and actions on categories

Definition. (Deligne, 1997)

An action T of a monoid $\mathbf{M}$ on categories is specified by

- a category $\mathbf{C} = T(\bullet)$
- an endofunctor $T(u) : \mathbf{C} \to \mathbf{C}$, for every element u of $\mathbf{M}$,
- natural isomorphisms $T_{u,v} : T(u)T(v) \Rightarrow T(uv)$ and $T_\bullet : 1_{\mathbf{C}} \Rightarrow T(1)$

satisfying the following coherence conditions:



Coherent presentations and actions on categories

Theorem. [Gaussent-Guiraud-M., 2013]

Let $\mathbf{M}$ *be a monoid and let* Σ *be a coherent presentation of* $\mathbf{M}$*.*
There is an equivalence of categories

$$\mathrm{Act}(\mathbf{M}) \approx 2\mathrm{Cat}(\Sigma_2^\top/\Sigma_3, \mathsf{Cat})$$

▶ Such equivalence for the Garside's presentation of spherical Artin monoids (Deligne, 1997)

Consequence.

To determine an action of an Artin group $\mathbf{B}(\mathbf{W})$ on a category $\mathbf{C}$, it suffices to attach to any generating 1-cell $s \in S$ and endofunctor $T(s) : \mathbf{C} \to \mathbf{C}$, to any generating 2-cell an isomorphism of functors such that these satisfy coherence Zamolochikov relations.

Other applications

- Coherent presentation of **Garside** monoids [Gaussent-Guiraud-Malbos, 2013].

- Coherent presentation of **plactic** and **Chinese** monoids [Guiraud-Malbos-Mimram, 2013].