Realization of differential graded Lie algebras …or L-infinity algebras

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Stasheff

Sullivan

Tanré

Thomas

A brief introduction

Once upon a time... 

Category of simplicial sets

$$\mathbf{DGL} \underset{\lambda}{\overset{\langle \cdot \rangle}{\rightleftarrows}} \mathbf{SSet}$$

Category of differential graded (over $\mathbb{Z}$) Lie algebras over a field of characteristic zero

Theorem

With the appropriate restrictions, the induced functors in the homotopy categories are equivalences.

$$Ho\,\mathbf{DGL}_1^{\mathbb{Q}} \xrightarrow{\cong} Ho\,\mathbf{SSet}_1^{\mathbb{Q}}$$

$$Ho\,\mathbf{DGL}_{\mathcal{N},f}^{\mathbb{Q}} \xrightarrow{\cong} Ho\,\mathbf{SSet}_{\mathcal{N},f}^{\mathbb{Q}}$$

The *"unrestricted"* situation is also useful...

Example 1: mapping spaces...

Let X be a finite nilpotent CW-complex.

Let Y be a finite type nilpotent CW-complex.

Let L be a DGL model of Y.

Let C be a finite dimensional coalgebra model of X.

In $Hom(C, L)$ define:

$$[g, h] : C \longrightarrow L,$$

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{g \otimes h} L \otimes L \xrightarrow{[\,,\,]} L$$

$$Dg = d \circ g + (-1)^{|g|} g \circ d \,.$$

Theorem $Hom(C, L)$ is a DGL model of map(X, Y).

Consequences...

Let Y be a rational simply connected complex and $\alpha \in \pi_n(Y)$. Then,

$$\pi_* \mathrm{map}_\alpha(S^n, Y) \otimes \mathbb{Q} \cong \ker ad_\alpha \oplus \mathrm{coker}\ ad_\alpha$$

$$\pi_* \mathrm{map}^*_\alpha(S^n, Y) \otimes \mathbb{Q} \cong \pi_*(Y) \otimes \mathbb{Q}$$

If $\pi_*(Y) \otimes \mathbb{Q}$ is an infinitely generated Lie algebra, then $\pi_* \mathrm{map}_f(X, Y)$ is also infinitely generated.

Example 2: Deformations…

"…Every deformation functor is governed by a differential graded Lie algebra…"

Let A be an associative algebra.

$$A\big[[t]\big] = \{\sum_{n\geq 0} f_n t^n, \quad f_n \in A\}$$

A *deformation* of the multiplication in A is a bilinear map

$$*\colon A\big[[t]\big] \times A\big[[t]\big] \longrightarrow A\big[[t]\big]$$

$$\sum_{n\geq 0} f_n t^n * \sum_{n\geq 0} g_n t^n = \sum_{n\geq 0} h_n t^n, \qquad h_0 = f_0 g_0.$$

Two deformations $*$ y $*'$ of the multiplication in A are equivalent if there is an automorphism

$$\varphi\colon A\big[[t]\big] \xrightarrow{\cong} A\big[[t]\big]$$

which fix the constant term of each series and

$$\varphi(f * g) = \varphi(f) *' \varphi(g), \qquad f, g \in A.$$

Let R be a local commutative algebra with maximal ideal $\mathfrak{M}$ ($R/\mathfrak{M} = k$ the coeffcient field)

Let A be an associative algebra. An R-deformation of A is an R-bilinear associative map

$$*\colon (A \otimes R) \otimes_R (A \otimes R) \longrightarrow A \otimes R$$

such that, modulo $\mathfrak{M}$, it reduces to the product of A.

$Def(A;R)$ =Set of equivalence classes of R-deformations of A.

There exists a DGL L for which,

$$Def(A;R) \cong \widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\sim$$

Let L be a DGL. $z \in L_{-1}$ is a Maurer-Cartan element if

$$\partial z = -\tfrac{1}{2}[z,z].$$

We will denote the set of Maurer-Cartan elements in L by $\mathrm{MC}(L)$.

DGL responsible for associative deformations

Let $C^*(A,A)$ be the Hochschild complex of A endowed with the Gerstenhaber bracket.

$$C^p(A,A) = Hom(A^{\otimes p}, A) \qquad d\colon C^p(A,A) \longrightarrow C^{p+1}(A,A)$$

$$df(a_0,\ldots,a_p) = (-1)^p a_0 f(a_1,\ldots,a_p) - f(a_0,\ldots,a_{p-1})\, a_p$$
$$- \Sigma_{i=0}^{p-1}(-1)^{i+p} f(a_0,\ldots,a_i a_{i+1},\ldots,a_p)$$

$$f \in C^p(A,A),\ g \in C^q(A,A),\ f \bullet g \in C^{p+q-1}(A,A)$$

$$(f \bullet g)(a_1,\ldots,a_{p+q-1}) =$$

$$\Sigma_{i=1}^{p}(-1)^{i(q+1)} f\Big(a_1,\ldots,a_i, g(a_{i+1},\ldots,a_{i+q}), a_{i+q+1},\ldots,a_{p+q-1}\Big)$$

$$[f,g] = f \bullet g - (-1)^{pq} g \bullet f$$

Desuspend $C^*(A, A)$ to obtain a differential graded Lie algebra concentrated in degrees ≤ 1:

$$\mathcal{L}^{As} = \mathcal{L}^{As}_{\leq 1} \qquad \mathcal{L}^{As}_{-p} = C^{p+1}(A, A)$$

Given $*$ a deformation of A, write:

$$a * b = ab + B(a, b)$$

$$B\colon A \otimes A \longrightarrow A \otimes \mathfrak{M} \qquad B \in C^2(A, A) \otimes \mathfrak{M}$$

Associativity of $*$ translates to $\quad dB = -\frac{1}{2}[B, B]$

In other words, z is a Maurer-Cartan element of $\mathcal{L}^{As} \otimes \mathfrak{M}$.

It turns out that two R-deformations, $*$ and $*'$ are equivalent if and only if the corresponding Maurer-Cartan elements z and z' in the DGL $\mathcal{L}^{As} \otimes \mathfrak{M}$ are gauge equivalent.

$Def(A; R)$ is controlled by $\mathcal{L}^{As} \otimes \mathfrak{M}$, that is,

$$Def(A; R) \cong \widetilde{\mathsf{MC}}(\mathcal{L}^{As} \otimes \mathfrak{M}).$$

Goal

Develop a consistent *homotopy theory* in (unbounded!) $\mathbf{DGL}$, or more generally in $\mathbf{L}_\infty$, to be able to algebraically model the homotopy type of non-connected spaces.

L_∞ algebras

$$(L, \{\ell_k\}_{k\geq 1})$$

$L = \oplus_{n\in\mathbb{Z}} L_n$ $\qquad$ $\ell_k = [\,,\dots,\,]\colon \otimes^k L \to L$ $\qquad$ of degree $k-2$

(1) Skew-symmetry: $[x_{\sigma(1)},\dots,x_{\sigma(k)}] = \pm[x_1,\dots,x_k]$.

(2) Jacobi identities:

$$\sum_{i+j=n+1}\ \sum_{\sigma\in S(i,n-i)} \pm\Big[[x_{\sigma(1)},\dots,x_{\sigma(i)}],x_{\sigma(i+1)},\dots,x_{\sigma(n)}\Big] = 0.$$

$\ell_1 = \partial$ is a differential in L.

$\partial[a,b] = [\partial a, b] + (-1)^{|a|}[a,\partial b]$.

$$\pm\Big[a,[b,c]\Big] \pm \Big[b,[c,a]\Big] \pm \Big[c,[a,b]\Big] =$$

$$\pm\partial[a,b,c] \pm [\partial a,b,c] \pm [a,\partial b,c] \pm [a,b,\partial c].$$

....

A DGL is an L_∞ algebra for which $\ell_k = 0$, $k \geq 3$.

Cochain and realization functor

Category of commutative differential graded algebras

Cochain:

$$\mathcal{C}^\infty \colon \mathbf{L}_\infty \rightsquigarrow \mathbf{CDGA}$$

$$\mathcal{C}^\infty(L) = (\Lambda(sL)^\sharp, d) = (\Lambda V, d)$$

in which d is defined according to the following pairing:

$$d = \sum_{j \geq 1} d_j \qquad d_j V \subset \Lambda^j V$$

$$\langle d_j v; sx_1 \wedge \cdots \wedge sx_j \rangle = \pm \langle v; s[x_1, \dots, x_j] \rangle$$

Realization:

$$\mathcal{C}^\infty \colon \mathbf{L}_\infty \rightsquigarrow \mathbf{CDGA}, \qquad \langle \cdot \rangle \colon \mathbf{CDGA} \rightsquigarrow \mathbf{SSet}$$

$$\langle \cdot \rangle \colon \mathbf{L}_\infty \rightsquigarrow \mathbf{SSet}$$

Points, augmentations and Maurer-Cartan elements

In $\mathbf{CDGA}$ points correspond to augmentations:

$$* \hookrightarrow X$$

$$A(\mathbb{Q}) \twoheadleftarrow A(X)$$

In $\mathbf{DGL}$ or $\mathbf{L}_\infty$ points correspond to *Maurer-Cartan* elements:

Let L be a DGL. $z \in L_{-1}$ is a Maurer-Cartan element if

$$\partial z = -\frac{1}{2}[z, z].$$

Let L be an L_∞ algebra. $z \in L_{-1}$ is a Maurer-Cartan element if

$$\sum_{k=1}^{\infty} \frac{1}{k!}[z, \overset{k}{\ldots}, z] = 0.$$

Theorem $\quad \mathrm{MC}(L) \cong \mathit{Aug}\, \mathcal{C}^\infty(L)$

Homotopies, cylinders and the Lawrence-Sullivan construction

In the classical 1-connected or nilpotent category, the notion of homotopy in the algebraic categories is well known and fully understood.

How to extended it to the non-connected case?

$f, g\colon (X, x_0) \longrightarrow (Y, y_0)$ are homotopic (in the based category!) if there is a based homotopy

$$H\colon (X \times I, \{x_0\} \times I) \longrightarrow (Y, y_0)$$

between f and g.

$H\colon X \wedge I^+ \longrightarrow Y$

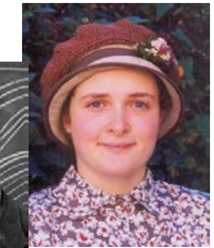
H corresponds to a based map

$$H\colon I^+ \longrightarrow \mathrm{map}(X, Y),$$

$$H(+) = c_{y_0},\ H(0) = f,\ H(1) = g.$$

Hence, a good cylinder in $\mathbf{DGL}$ should be a candidate for a model of I^+

"A free differential Lie algebra for the interval"



$$\mathfrak{L} = (\widehat{\mathbb{L}}(a, b, x), \partial)$$

$\widehat{\mathbb{L}}$ denotes *complete free Lie algebra*.

a and b are Maurer-Cartan elements.

i^{th} Bernoulli number

$$\partial x = [x, b] + \sum_{i=0}^{\infty} \frac{B_i}{i!} \operatorname{ad}_x^i (b - a),$$

Definition

Let L be an L_∞ algebra. Two Maurer-Cartan elements $z_0, z_1 \in \mathrm{MC}(L)$ are *homotopic*

$$z_0 \sim z_1$$

if there is an L_∞ morphism

$$\phi \colon \mathfrak{L} \to L$$

such that $\mathrm{MC}(\phi)(a) = z_0$ and $\mathrm{MC}(\phi)(b) = z_1$.

$$\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\sim$$

Localization, components and realization

Let L be an $L\infty$ algebra and $z \in \mathsf{MC}(L)$.

The *perturbation of the bracket* ℓ_k *by* z is the new bracket

$$\ell_k^z(x_1, \ldots, x_k) = [x_1, \ldots, x_k]_z = \sum_{i=0}^{\infty} \frac{1}{i!} \ell_{i+k}(z, \overset{i}{\ldots}, z, x_1, \ldots, x_k).$$

$(L, \{\ell_k^z\})$ is again an L_∞ algebra.

The *localization of* L *at* z is the new L_∞ algebra $L^{(z)}$:

$$L_i^{(z)} = \begin{cases} L_i & \text{if } i > 0, \\ \ker \ell_1^z & \text{if } i = 0, \\ 0 & \text{if } i < 0. \end{cases}$$

The brackets are induced by ℓ_k^z for any $k \geq 1$.

Theorem

Let L be an L_∞ algebra. Then,

$$\langle L\rangle \simeq \dot{\bigcup}_{z\in\widetilde{\mathsf{MC}}(L)} \langle L^{(z)}\rangle.$$

Examples

The realization of the Lawrence-Sullivan construction has the homotopy type of S^0.

$$\langle \mathfrak{L}\rangle \simeq S^0$$

Let L the DGL which governs the infinitesimal deformations of the abelian Lie algebra of dimension n.

$$\langle L\rangle \simeq \dot{\cup}_{\mathbb{Z}}\, (S^1)^{n^2} \vee (\mathbb{C}P^\infty)^n.$$

Models of non-connected spaces

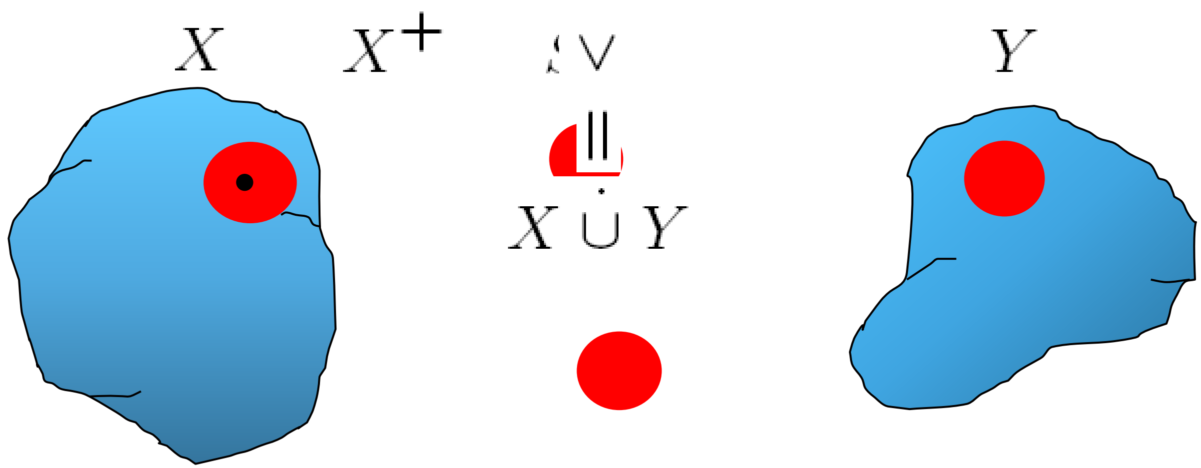
Input:

Family of nilpotent spaces of the homotopy type of finite type CW-complexes.

Family of 0-reduced DGL's of finite type.

Output:

DGL whose realization is of the homotopy type of the rationalization of the space whose components are the elements of the given family.



Given $L, M \in \mathbf{DGL}$ we denote by $L * M$ its coproduct. Recall that, given free presentations $L = \mathbb{L}(U)/I$, $M = \mathbb{L}(V)/J$, then $L * M = \mathbb{L}(U \oplus V)/\langle I, J\rangle$

Let L, M be non-negatively graded DGL's models of the path connected spaces X and Y.

Lemma $\mathbb{L}(u) * L$ is a model of $X \vee S^0$.

Perturb the differential by the Maurer-Cartan element u:

$$(\mathbb{L}(u) * L, \partial_u)$$

$$\partial_u(u) = \frac{1}{2}[u, u]$$

$$\partial_u a = \partial a + [u, a], \; a \in L$$

Nothing has changed! (except the base point)

Indeed, $\langle L\rangle \simeq \langle L_z\rangle$ for any $z \in \mathrm{MC}(L)$.

Lemma $(\mathbb{L}(u) * L, \partial_u)$ is a model of $X^+ = X \dot\cup \{\text{point}\}$.

Finally,

Theorem $(\mathbb{L}(u) * L * M, \partial_u * \partial_M)$ is a model of $X \dot\cup Y$.

More generally,

Theorem

Let X be a space with path components $\{Y, X_j\}_{j\in J}$ and let $\{L, L_j\}_{j\in J}$ be a family of non-negatively graded DGL's, each of which modeling the corresponding component. For each $j \in J$ consider the perturbed DGL

$$M_j = (\mathbb{L}(u_j) * L_j, \partial_{u_j}),$$

$$\partial_{u_j}(u_j) = \frac{1}{2}[u_j, u_j], \qquad \partial_{u_j} x = \partial_j x + [u_j, x], \qquad x \in L_j.$$

Then,

$$M = *_{j\in J} M_j * L$$

is a model of X.

Corollary

In particular, every non-connected space X has a free model generated by a vector space concentrated in degrees greater than or equal to -1.

Examples

$$(\mathbb{L}(u, x_n, y_1, y_3), \partial) \qquad \begin{aligned} \partial u &= \tfrac{1}{2}[u,u], \\ \partial x_n &= [u, x_n], \\ \partial y_1 &= 0, \quad \partial y_3 = [y_1, y_1], \end{aligned}$$

is a model of $S^n \,\dot{\cup}\, \mathbb{C}P^2$.

$$(\mathbb{L}(a_0, u_1, a_1, u_2, a_2, \ldots, u_n, a_n, \ldots), \partial) \quad \begin{aligned} \partial a_0 &= 0, \\ \partial u_i &= \tfrac{1}{2}[u_i, u_i], \\ \partial a_i &= [u_i, a_i], \quad i \geq 1, \end{aligned}$$

is a model of $\dot{\cup}_{n\geq 1} S^n$.

All of this in:

Urtzi Buijs, Aniceto Murillo, *Algebraic models of non connected spaces and homotopy theory of L_∞ algebras*, Adv. in Math. 236 (2013), 50-91.