A bridge between Chow and NChow, and algebraic geometry

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(A project in collaboration with Gonçalo Tabuada (MIT))

Chow groups

Let X be a smooth projective variety of dimension d over a field k. Var(k) the category of such varieties.

For a ring R (say R is $\mathbb{Z}$ or $\mathbb{Q}$) we can define the Chow Group

$CH^i(X)_R$ = *free R-module generated by i-codimensional cycles (closed subvarieties) of X, modulo rational equivalence.*

$Z_1 \sim_{rat} Z_2$ if and only if there exist $Z \subset \mathbb{P}^1 \times X$, and x_1, x_2 in $\mathbb{P}^1$ such that $Z_{|x_j \times X} = Z_j$.

We denote $CH^*(X)_R := \bigoplus_{i=0}^d CH^i(X)_R$ the (graded) Chow ring.

Example (1)

$CH^i(\mathbb{P}^n)_R = R$ for all $0 \leq i \leq d$ (a unit in each degree)

Example (2)

C curve, $CH^0(C)_R = R$, and $CH^1(C)_R = R \oplus A^1(C)_R$.

$A^1(C)_R$ is the group of cycles equivalent to zero by the relation: $Z_1 \sim_{alg} Z_2$ if and only if there exist a curve Γ, $Z \subset \Gamma \times C$, and x_1, x_2 in Γ such that $Z_{|x_j \times C} = Z_j$.

[Weil] $A^1(C)_{\mathbb{Z}}$ is a principally polarized abelian variety $J(C)$, the Jacobian of C.

For any X in Var(k), from $A^i(X)_{\mathbb{Q}} \subset CH^i(X)_{\mathbb{Q}}$, one can define an abelian variety $J^i(X)$ only up to isogeny (i.e. no natural polarization on it!).

GRR-Theorem

The Grothendieck group of X is

$$K_0(X)_R := K_0(\mathrm{Coh}(X))_R = K_0(\mathrm{D^b}(X))_R$$

Theorem (Grothendieck-Riemann-Roch)

The Chern character gives a functorial isomorphism $ch : K_0(X)_{\mathbb{Q}} \simeq CH^*(X)_{\mathbb{Q}}$.

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WARNING:

- ch does not naturally carry codimensional information.
- the isomorphism does not hold over $\mathbb{Z}$ in general.

Question 1: Rationality

X rational if a (Zariski) open of X is isomorphic to a (Zariski) open of $\mathbb{P}^d$.

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A very quick summary of known criteria:

- **[Lüroth, 1870's]**
  If $d = 1$, X is rational iff $X = \mathbb{P}^1$ equivalently $J^1(X) = 0$
- **[Castelnuovo-Enriques, 1920's]**
  If $d = 2$ and $k = \overline{k}$, cohomological criterion
- **[Clemens-Griffiths, 1972]**
  If $d = 3$, $k = \mathbb{C}$ and X rational, then
  $J^i(X) = 0$ for $i \neq 2$ and there is a curve Γ such that
  $J^2(X) \simeq J(\Gamma)$ as principally polarized abelian varitites.
- few, sparse cases for higher dimensions or different fields.

A very rough reason to care about derived categories

Clemens-Griffiths result is proved using resolution of singularities and the strong structure of principal polarization.

(WARNING - VERY ROUGH): if X is rational his invariants "behave like invariants of varieties of dimension $d-2$".
Notice: all abelian variety are ISOGENOUS to a Jacobian of a curve!!

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[Bondal-Orlov, ICM 2002] Do Semiorthogonal decompositions

$$\mathrm{D}^{\mathrm{b}}(X) = \langle T_1, \ldots, T_n \rangle$$

provide a way to do that? [B,O,Kapranov,Kuznetsov,Bolognesi,Auel,...]
[-,Bolognesi]: definition of categorical representability of X.

NOTE: The *unit* triangulated category is $\mathrm{D}^{\mathrm{b}}(\mathrm{Spec}(k))$.
$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^n)$ is decomposed by units.

Question 2: A Conjecture

Let X be in $\mathbb{P}^n$ a complete intersection of degree $(d_1, \ldots, d_l)$.

Conjecture

There exists an integer $\kappa = \kappa(n, d_j)$ such that $CH_i(X)_{\mathbb{Q}} = \mathbb{Q}$ for $i < \kappa$.

Roughly: very small $\mathbb{Q}$-cycles on X behave as in the projective space.

[Esnault-Levine-Viehweg 1997] Provide a bound of order $\frac{\kappa}{d_1}$.

Commutative Motives [Grothendieck]

IDEA: cycles on $X \times Y$ = correspondences $CH^*(X)_R \to CH^*(Y)_R$.

The category Chow$(k)_R$ *has (up to taking pseudo abelian envelope):*
Objects are triples (X, p, i)*:*
$X \in \mathrm{Var}(k)$, $p \in CH^d(X \times X)_R$ *such that* $p^2 = p$.
Morphisms $\mathrm{Hom}((X, p, i), (Y, q, j))$ *are cycles in* $CH^{d+i-j}(X \times Y)_R$*, composed with* p *and* q.

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There is a universal functor: $h_R : \mathrm{Var}(k) \to \mathrm{Chow}(k)_R$:

$h(X)_R = (X, \Delta, 0)$ (Note: $\Delta = \mathrm{id}$)

The unit is $\mathbf{1}_R = (\mathrm{Spec}(k), \mathrm{id}, 0) = h_R(\mathrm{Spec}(k))$

The Lefschetz motive is $\mathbf{L}_R = (\mathrm{Spec}(k), \mathrm{id}, -1)$

Examples

Example (1)

$h_R(\mathbb{P}^n) = \mathbf{1}_R \oplus \ldots \oplus \mathbf{L}_R^n$

Example (2)

$h_{\mathbb{Q}}(C) = \mathbf{1}_{\mathbb{Q}} \oplus h^1(C)_{\mathbb{Q}} \oplus \mathbf{L}_{\mathbb{Q}}$

The submotive $h^1(C)_{\mathbb{Q}}$ identifies $J(C)$ up to isogeny.

$$\mathrm{Hom}(h^1(C)_{\mathbb{Q}}, h^1(C')_{\mathbb{Q}}) = \mathrm{Hom}(J(C), J(C')) \otimes \mathbb{Q}$$

Motives and abelian varieties

The submotive $h^1(C)_{\mathbb{Q}}$ identifies $J(C)$ up to isogeny. Since any abelian variety is isogenous to a Jacobian of a curve, we have

$$\mathrm{Ab}(k)_{\mathbb{Q}} \subset \mathsf{Chow}(k)_{\mathbb{Q}}$$

This inclusion factors through a stronger equivalence (numerical equivalence) giving

$$\mathrm{Ab}(k)_{\mathbb{Q}} \subset \mathsf{Num}(k)_{\mathbb{Q}}$$

Noncommutative motives [Kontsevich]

The category $\mathsf{NChow}(k)_R$ *has (up to taking pseudo abelian envelope)*

Objects: smooth, proper dg-categories over k

Morphisms: $\mathrm{Hom}(A, B) = K_0(A^{op} \otimes B)_R$

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Noncommutative motives [Kontsevich]

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Theorem (Tabuada)

There is a universal additive functor $U_R : \mathrm{dgcat}(k) \to \mathrm{NChow}(k)_R$ *s.t.:*

- *Any additive invariant factors through* U_R
- *if* $A = \langle T, T^{\perp} \rangle$, *then* $U_R(A) = U_R(T) \oplus U_R(T^{\perp})$

Moreover one can define a (semisimple) category $\mathrm{NNum}(k)_R$ of Numerical noncommutative motives.

The bridge

1st Problem. Noncommutative motives do not know codimension!

To build a bridge from $\mathsf{Chow}(k)_R$ to $\mathsf{NChow}(k)_R$, we have first to "kill Lefschetz":

Consider the orbit category $\mathsf{Chow}(k)_R / - \otimes \mathbb{L}_R$.

2nd Problem. How to connect $\mathsf{Var}(k)$ and $\mathsf{dgcat}(k)$?

[Lunts-Orlov] The category $\mathrm{D^b}(X)$ can be uniquely enhanced into a (smooth projective) dg-category $\mathrm{D^b_{dg}}(X)$.

The Bridge

[Tabuada] We have the following diagram

$$
\begin{array}{ccc}
\mathrm{Var}(k) & \xrightarrow{\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(-)} & \mathrm{dgcat}(k) \\
\downarrow{\scriptstyle h_R} & & \downarrow{\scriptstyle U_R} \\
\mathrm{Chow}(k)_R/-\otimes\mathbb{L} & & \mathrm{NChow}(k)_R \\
\downarrow & & \downarrow \\
\mathrm{Num}(k)_R/-\otimes\mathbb{L} & & \mathrm{NNum}(k)_R
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\downarrow h_{\mathbb{Q}} & & \downarrow U_{\mathbb{Q}} \\
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\downarrow & & \downarrow \\
\mathrm{Num}(k)_{\mathbb{Q}}/-\otimes\mathbb{L} & \xrightarrow{R_N} & \mathrm{NNum}(k)_{\mathbb{Q}}
\end{array}$$

R and R_N are fully faithful. All these respect units.
Proof based on GRR

From now on, $k = \overline{k}$ has char=0.

Definition (Marcolli-Tabuada)

For A in dgcat(k), the Jacobian $\mathbf{J}(A)$ is defined as the component sitting in $R_N(\mathrm{Ab}(k)_{\mathbb{Q}})$ of the nuerical NC-motive of A.
It is well-defined up to isogeny.

Theorem (Marcolli-Tabuada)

if $A = \mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(X)$ and X satisfies Grothendieck conjectures, then (up to isogeny)

$$\mathbf{J}(X) = \bigoplus_{i=0}^{d} J^i(X)$$

From Semiorthogonal decompositions to Jacobians

Theorem (-,Tabuada)

If $\mathrm{D}^{\mathrm{b}}(X) = \langle T_X, T_X^{\perp} \rangle$, *and* $\mathrm{D}^{\mathrm{b}}(Y) = \langle T_Y, T_Y^{\perp} \rangle$, *and* $T_Y \simeq T_X$ *(as dg-categories) then*

- *there is a map* $\tau : \oplus J^i(X) \to \oplus J^i(Y)$,
- τ *injective if* $\mathbf{J}(T_X^{\perp}) = 0$,
- τ *surjective if* $\mathbf{J}(T_Y^{\perp}) = 0$

Moreover, one can give conditions under which τ *preserves principal polarizations (if defined).*

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Example (Bondal-Orlov=Clemens-Griffiths for Fano threefolds)

If X *is a Fano threefold, there is often a semiorthogonal decomposition with a component resconstructing the intermediate Jacobian as principally polarized abelian variety.*

Chow groups of intersections of quadrics

Theorem (-,Tabuada)

Let X be a complete intersection of type $(2,2)$ or of type $(2,2,2)$ and odd dimension. Then $CH_i(X)_{\mathbb{Q}} = \mathbb{Q}$ for all $i < \kappa$.

IDEA OF PROOF: $\mathrm{D^b}(X) = \langle \mathrm{D^b}(k), \ldots, \mathrm{D^b}(k), T \rangle$, with one of the following:

- T splits into units,
- $T = \mathrm{D^b}(C)$, for a curve C
- $T =$ category of modules over a Clifford algebra

In any case we can separate the "non-unital" part and count the units to get the required number of units in $U(\mathrm{D^b_{dg}}(X))$, hence in $CH^*(X)_{\mathbb{Q}}$.