The Milnor-Thurston determinant and the Ruelle transfer operator

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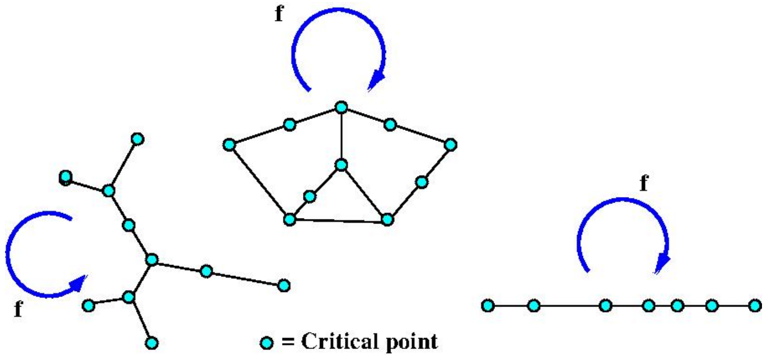
Motivation and set-up

We consider (I, f), a piecewise continuous and strictly monotone map of a 1 dimensional space. We may take I to be an Interval:



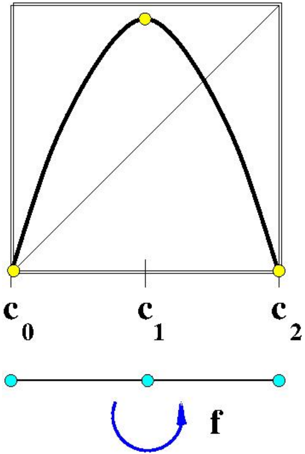
Analytic structures related to $h_{\mathrm{top}}(f)$

The dynamical system (I, f) has a topological entropy $h_{\mathrm{top}}(f)$. We are interested in related analytic structures. Here is the zoo:

- **$\mathbf{L}(t)$: Lap number generating function.**
- $D(t)$: Milnor-Thurston kneading determinant.
- $\zeta_{AM}(t)$: Artin-Mazur topological zeta-function.

- $\mathcal{L}$: Ruelle transfer operator for (I, f).
- $\zeta_R(t)$: Ruelle dynamical zeta-function.

Backward iterates of critical points



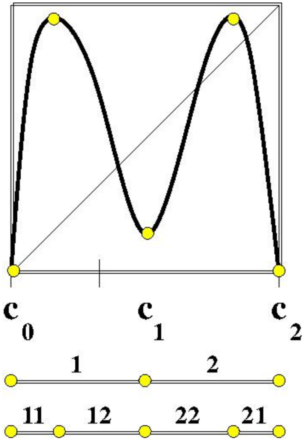
$$\mathrm{Crit}(f) = \{c_0, c_1, c_2\}$$

"Partition" into open intervals:

$$I_1 = (c_0, c_1), \; I_2 = (c_1, c_2)$$

$$Z_1 = \{I_1, I_2\}$$

Backward iterates of critical points



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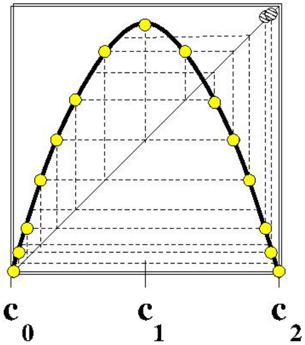
$$Z_1 = \{I_1, I_2\}$$

Refinement by backward iteration of critical points:

$$Z_2 = f^{-1} Z_1 \vee Z_1$$

$$Z_2 = \{I_{11}, I_{12}, I_{22}, I_{21}\}$$

Backward iterates of critical points



$$Z_n = f^{-(n-1)} Z_1 \vee \cdots \vee Z_1$$

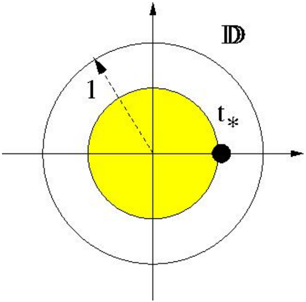
Misiurewicz and Szlenk:

$$h_{\mathrm{top}} = \lim_{n\to\infty} \frac{1}{n} \log \# Z_n$$

Lap number generating function:

$$L(t) = \sum_{n\geq 0} t^n \, \# Z_n \in \mathbb{Z}_+[[t]]$$

An a priori simple analytic structure of the Lap generating function, example:



$$L(t) = 1 + 2t + 4t^2 + 8t^3 + 14t^4 + \dots$$

Coefficients are non-negative integers.

- L analytic for: $|t| < t_* = e^{-h_{\text{top}}}$
- L diverges for $t > t_* = e^{-h_{\text{top}}}$

Analytic structures related to $h_{\mathrm{top}}(f)$

Returning to the zoo:

- $L(t)$: Lap number generating function.
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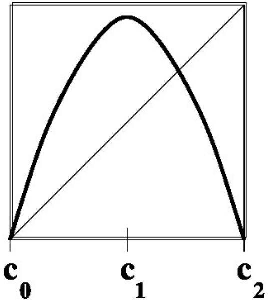
Dual picture: Forward iterates of critical points



$$c_0 < c_1 < \cdots c_d < c_{d+1}.$$

Intervals of monot.: $I_k = (c_k, c_{k+1})$,

$$f_k : I_k \to I = (c_0, c_{d+1})$$

is strictly monotone and continuous.
Need not be defined at c_k and c_{k+1}.

But $f(c_k^+)$ and $f(c_{k+1}^-)$ are well-defined.

Introduce "directed" points to keep track of directed limits:

$$x^+ = (x, +1), \; x^- = (x, -1)$$

Lifting the map to a directed point map

A directed point $\widehat{x}$ denotes either $(x,+1)$ or $(x,-1)$ (limit from the right/left).

On each directed interval $[c_k^+, c_{k+1}^-]$ the map either preserves or reverses orientation, also at endpoints. Set:

- $s(f,\widehat{x}) = +1$ if f preserves the orientation at $\widehat{x}$,
- $s(f,\widehat{x}) = -1$ if f reverses the orientation at $\widehat{x}$.

We "lift" f to a map on the space of directed points:

$$\widehat{f}(\widehat{x}) = \widehat{f}((x,\epsilon)) = \left(\lim_{t\to 0^+} f(x+\epsilon t), \;\; s(f,\widehat{x})\, \epsilon \right).$$

Kneading invariant

When $x < y$, declare $x < x^+ < y^- < y$ and for $\widehat{x} \in \widehat{I}$ and $u \in \mathbb{R}$:

$$\sigma(\widehat{x}, u) = \begin{cases} +1/2 & \text{if } \widehat{x} < u \\ -1/2 & \text{if } \widehat{x} > u \end{cases}$$

and for $c \in \mathrm{Crit}(f)$ the kneading invariant (coefficients $= \pm\frac{1}{2}$):

$$\theta_c(\widehat{x}, t) = \sum_{n \geq 0} t^n \; s(f^n, \widehat{x}) \; \sigma(\widehat{f^n}(\widehat{x}), c).$$

The kneading "determinant" (in our unimodal case):

$$D(t) = \theta_{c_1}(c_1^+, t) - \theta_{c_1}(c_1^-, t).$$

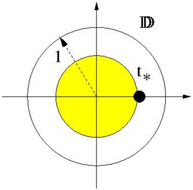
Recall: The lap generating function

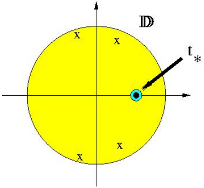
$$L(t) = 1 + 2t + 4t^2 + 8t^3 + 14t^4 + \ldots$$

diverges for $t > t_* = e^{-h_{\mathrm{top}}}$
while $D(t)$ is analytic in $\mathbb{D} = \{|t| < 1\}$
since coefficients are in $\{-1, 0, 1\}$.

Cancellations of backward and forward orbit contributions: $D(t) \times L(t)$ is analytic in $\mathbb{D}$ with no roots!

- $e^{-h_{\mathrm{top}}}$ is a pole of $L(t)$.
- $e^{-h_{\mathrm{top}}}$ is the smallest root of $D(t)$.





With d critical points, $\mathrm{Crit}(f) = \{c_0, ..., c_{d+1}\}$ one may introduce a $(d+1) \times (d+1)$ kneading matrix:

$$R_{jk}(t) = \begin{cases} \theta_{c_k}(c_0^+, t) + \theta_{c_k}(c_{d+1}^-, t) \quad , & j = 0 \\ \theta_{c_k}(c_j^+, t) - \theta_{c_k}(c_j^-, t) \quad , & 1 \le j \le d \end{cases}$$

and a Milnor-Thurston kneading determinant:

$$D(t) = \det R_{jk}(t).$$

Again a magic property (much harder to prove): $D(t)L(t)$ is analytic in $\mathbb{D}$ and has no roots for $|t| < e^{-h_{\rm top}}$. Once again:

- $t_* = e^{-h_{\rm top}}$ is the smallest root of $D(t)$.

Analytic structures related to $h_{\rm top}(f)$

Back at the zoo ...:

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- $\mathcal{L}$: Ruelle transfer operator for (I, f).
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The Artin-Mazur topological zeta-function (tacitly assuming finitely many fixed points):

$$\zeta_{AM}(t) = \exp\left(\sum_{n\geq 1} \frac{t^n}{n} \#\mathrm{Fix}(f^n)\right).$$

Yet again magic cancellations:

$$D(t) \times \zeta_{AM}(t)$$

is analytic in $\mathbb{D}$ with no roots!

Analytic structures related to $h_{\text{top}}(f)$

Continuing the tour at the zoo ...:

- $L(t)$: Lap number generating function.
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The Ruelle transfer operator and zeta-function

For $\phi \in BV(I)$, a function of bounded variation on I, we set:

$$\mathcal{L}\phi(y) = \sum_{x:f(x)=y} \phi(x)$$

Acting on the constant function we simply count pre-images:

$$\mathrm{Card}\{x : f^n(x) = y\} = \mathcal{L}^n\, \mathbf{1}(y).$$

One has:

- $r_{\rm sp}(\mathcal{L}) = \lim_{n\to\infty} \|\mathcal{L}^n\|_{BV}^{1/n} = e^{h_{\rm top}}$.
- $\mathcal{L}$ is a positive operator $\Rightarrow (r_{\rm sp}(\mathcal{L}) - \mathcal{L})$ is non-invertible.

Baladi and Keller (1990) defines a zeta-function (when f is expanding):

$$\zeta_R(t) = \exp\left(\sum_{n\geq 1} \frac{t^n}{n} \#\mathrm{Fix}(f^n)\right).$$

They show that on the space of BV-functions:

$$(\zeta_R(t))^{-1} = \det(1 - t\mathcal{L})$$

"det" is a "dynamical" determinant introduced by Ruelle.
Both functions are analytic in $\mathbb{D}$. Zeros are in 1-1 correspondance with the reciprocal of the eigenvalues of $\mathcal{L}$, greater than 1 in absolute value.

- Now note that $\zeta_R(t) = \zeta_{AM}(t)$.

So ($\sim$ = analytic, same roots in $\mathbb{D}$):

$$D(t) \sim \zeta_{AM}(t)^{-1} = \zeta_R(t)^{-1} \sim \det(1 - t\mathcal{L}).$$

$D(t)$ not only determines $h_{\rm top}$ but also describes eigenvalues of $\mathcal{L}$!

- $D(t)$ is determined by forward orbits
- while $\mathcal{L}$ uses backward iteration.

A thought: Since $\mathcal{L}$ is based upon backward iterates of f, perhaps the dual operator $\mathcal{L}'$ should use forward iterations by f?

Quest starting in the 90s ...:
Express $D(t)$ as a determinant of the dual Ruelle operator.

Several partial results: Baladi and Ruelle (1994), ..., Gouëzel (2001). Calculations use BV-functions, are indirect and difficult. They do not quite cover the original problem.

Reason:

- BV function space is too large $\Rightarrow$ The dual space is too small.

Is it possible to tailor a Banach space better to our needs...?

Let $\mathcal{S}$ be the space of piecewise constant functions on (c_0, c_{d+1}) and X the closure of $\mathcal{S}$ in BV. X' denotes the dual space.

Theorem (R. 2015)

$D(t)$ equals a (regularized) determinant of $\mathcal{L}'$ acting upon X'. This (regularized) determinant is analytic in $\mathbb{D}$.

Corollary

$h_{\mathrm{top}} > 0 \Rightarrow e^{-h_{\mathrm{top}}}$ is the smallest zero of $D(t)$.

Proof:

- $r_{\mathrm{sp}}(\mathcal{L}') = r_{\mathrm{sp}}(\mathcal{L}) = e^{-h_{\mathrm{top}}}$.
- $\mathcal{L}$ is a **positive** operator $\Rightarrow e^{h_{\mathrm{top}}} \in$ spectrum of $\mathcal{L}$ (and of $\mathcal{L}'$).
- For $t \in \mathbb{D}$ we have: $D(t) = 0 \Leftrightarrow 1/t$ is an eigenvalue of $\mathcal{L}'$

Functions of bounded variation

$\phi : I = (c_0, c_{d+1}) \to \mathbb{C}$ is said to be of bounded variation (BV) iff

$$\mathrm{var}\,\phi = \sup\{\sum_{i=1}^{N-1} |\phi(x_i) - \phi(x_{i+1}| : c_0 < x_1 < \ldots < x_N < c_{d+1}\} < +\infty$$

When $\mathrm{var}\,\phi < +\infty$ we have existence of right and left limits:

- $\phi(x^+) = \lim_{t \to 0^+} \phi(x + t)$ for $c_0 \leq x < c_{d+1}$,
- $\phi(x^-) = \lim_{t \to 0^+} \phi(x - t)$ for $c_0 < x \leq c_{d+1}$.

We introduce the "boundary" value of ϕ:

$$\partial\phi = \phi(c_0^+) + \phi(c_{d+1}^-)$$

We define the BV norm:

$$\|\phi\| = \|\phi\|_{\rm BV} = {\rm var}\,\phi + |\partial\phi|.$$

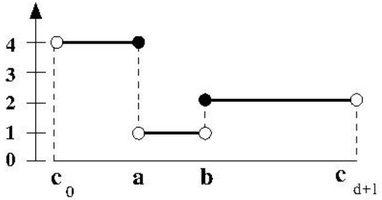
Denote by $\mathcal{S}$ the piecewise constant functions on I and let X be the completion of $\mathcal{S}$ w.r.t $\|\cdot\|$ (same as closure in BV).

Every $\phi \in \mathcal{S}$ may be written as $\sum_{\text{finite}} w_i \sigma_{\widehat{u}_i}$ in the basis:

$$\sigma_{\widehat{u}}(x) = \begin{cases} +1/2 & \text{if } \ \widehat{u} < x \\ -1/2 & \text{if } \ \widehat{u} > x \end{cases}, \qquad \widehat{u} \in \widehat{I} = [c_0^+, c_{d+1}^-).$$

Example:

$$\phi = 3\sigma_{c_0^+} - 3\sigma_{a^+} + \sigma_{b^-}$$



The BV-norm of ϕ:

$$\|\phi\| = 3 + 3 + 1 = 4$$

.

For $\phi = \sum_{\text{finite}} w_i \sigma_{\widehat{u}_i}$ we have: $\|\phi\| = \sum_i |w_i|$.

A represention of the dual space of X is obtained by acting upon the basis of X. For $\ell \in X'$ define:

$$\widehat{\ell}\,(\widehat{u}) = \langle\, \ell, \sigma_{\widehat{u}}\,\rangle, \qquad \widehat{u} \in \widehat{I}.$$

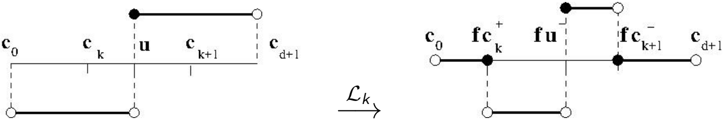
Then $\quad |\widehat{\ell}(\widehat{u})| \leq \|\ell\|_{X'} \quad$ and for $\quad \phi = \sum_{\rm finite} w_i \sigma_{\widehat{u}_i}$:

$$|\langle \ell, \phi\rangle| = |\sum_i w_i \widehat{\ell}(\widehat{u}_i)| \;\leq\; \|\phi\|_{\rm BV}\; \|\widehat{\ell}\|_\infty.$$

We have an isomorphism between X' and the bounded functions on the directed points of I with the uniform norm:

$$X' \quad \cong \quad B\,(\,[c_0^+, c_{d+1}^-)\,).$$

Example: Acting with $\mathcal{L}_k$, $1 \leq k \leq d+1$ upon σ_{u^-}:



The image is a linear combination of (at most) three basis functions. In the dual representation:

$$\begin{aligned} \widehat{\mathcal{L}}_k \widehat{\ell}(\widehat{u}) &= \langle \ell, \mathcal{L}_k \sigma_{\widehat{u}} \rangle \\ &= \mathbf{1}_{\widehat{I}_k}(\widehat{u})\ s(f,\widehat{u})\ \widehat{\ell}(\widehat{f}\widehat{u}) \\ &\quad -\sigma_{c_k}(\widehat{u})\ \ s(f,c_k^+)\ \ \widehat{\ell}(fc_k^+) \\ &\quad +\sigma_{c_{k+1}}(\widehat{u})\ \ s(f,c_{k+1}^-)\ \ \widehat{\ell}(fc_{k+1}^-) \end{aligned}$$

Adding terms: $\widehat{\mathcal{L}} = \sum_{k=1}^{d+1} \widehat{\mathcal{L}}_k$ we get:

$$\widehat{\mathcal{L}} = S - PS$$

in which we find a signed Koopman operator of norm $\|S\| = 1$:

$$S\,\widehat{\ell}\,(\widehat{u}) = s(f,\widehat{u})\,\ell(\widehat{f}\,\widehat{u})$$

and a finite rank projection operator, $P^2 = P$:

$$\begin{aligned} P\,\widehat{\ell}\,(\widehat{u}) &= \sigma_{c_0}(\widehat{u})\ \left(\widehat{\ell}\,(c_0^+) + \widehat{\ell}\,(c_{d+1}^-)\right) \\ &+ \sum_{k=1}^{d} \sigma_{c_k}(\widehat{u})\ \left(\widehat{\ell}\,(c_k^+) - \widehat{\ell}\,(c_k^-)\right) \end{aligned}$$

Good news: $\widehat{\mathcal{L}} = S - PS$ is a finite $(d+1)$ rank pertubation of the signed Koopman operator of norm 1. For $t = 1/\lambda \in \mathbb{D}$, $(1 - tS)$ is invertible:

$$(1 - tS)^{-1} = 1 + tS + t^2 S^2 + \cdots$$

To find eigenvalues with $|\lambda| > 1$ of $\widehat{\mathcal{L}}$ is equivalent to finding values of $t = 1/\lambda \in \mathbb{D}$ for which:

$$\begin{array}{rcl} & 1 - t\mathcal{L} & \text{is non-invertible} \\ = & (1 - tS) + tPS & \\ = & (1 + tPS(1 - tS)^{-1})(1 - tS) & \\ \Leftrightarrow & 1 + tPS(1 - tS)^{-1} & \text{is non-invertible} \end{array}$$

The projection P has finite rank so $(1 + tPS(1-tS)^{-1})$ is invertible iff it is invertible on

$$\mathrm{im}\, P = \mathrm{Span}\{\sigma_{c_0}, \ldots, \sigma_{c_d}\}$$

Let us compute the matrix elements on $\mathrm{im}\, P$ of

$$G(t) = P + tPS(1-tS)^{-1} = P(1-tS)^{-1}$$

Note first that

$$(1-tS)^{-1}\sigma_{c_k}(\widehat{u}) = (1 + tS + t^2S^2 + \cdots)\sigma_{c_k}(\widehat{u})$$
$$= \sum_{n \geq 0} t^n \; s(f^n, \widehat{u})\sigma(\widehat{f}^n(\widehat{u}), c) = \theta_{c_k}(\widehat{u}, t)$$

is nothing but the kneading coordinate of $\widehat{u}$ w.r.t. c_k.

Applying P to the result we get:

$$\begin{aligned} G(t)\,\sigma_{c_k}(\widehat{u}) &= P(1-tS)^{-1}\sigma_{c_k}(\widehat{u}) \\ &= \sigma_{c_0}(\widehat{u})\left(\theta_{c_k}(c_0^+,t)+\theta_{c_k}(c_{d+1}^-,t)\right) \\ &\quad + \sum_{j=1}^{d}\sigma_{c_j}(\widehat{u})\left(\theta_{c_k}(c_j^+,t)-\theta_{c_k}(c_j^-,t)\right) \\ &= \sum_{j=0}^{d}\sigma_{c_j}(\widehat{u})R_{jk}(t) \end{aligned}$$

where $(R_{jk}(t))$ is the Milnor-Thurston kneading matrix. It is non-invertible precisely when its determinant vanishes which is what we wanted to show.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