Matings and Thurston obstruction

in an early stage of a work in progress, since 1988

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Complex dynamics and quasi-conformal geometry
In memory of Tan Lei
Université d'Angers
Angers, October 23, 2017

Tan Lei (1963-2016)

Matings of polynomials

Douady's Bourbaki seminar 1983

Thurston's theorem,

Thurston 1983, Douady-Hubbard1993

Mary Rees, 1995 Proc. LMS

Unpublished manuscript 1986, Realization of matings of polynomials as rational maps of degree two

Tan Lei, 1986, C. R. Acad. Sci., Accouplements des polynômes quadratiques complexes

1992, Erg. Th. & Dynam. Sys. Matings of quadratic polynomilas

In 1986, I met Douady and Tan Lei in Paris

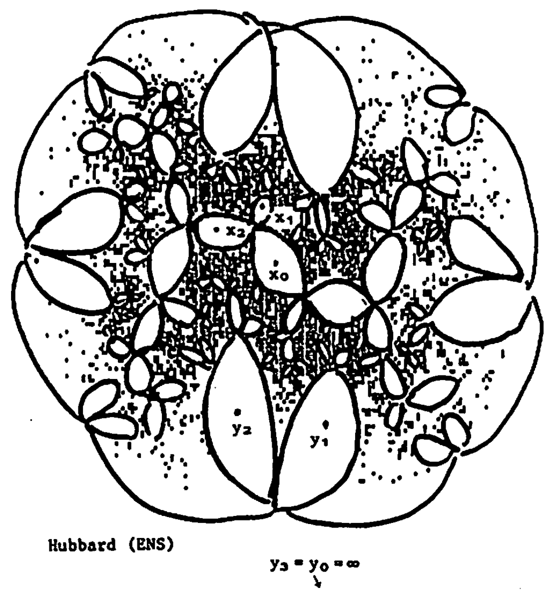
S. - Tan Lei, Max Planck Institute Preprint 1988, published 2000

Tan Lei, Cubic Newton's method, 1997

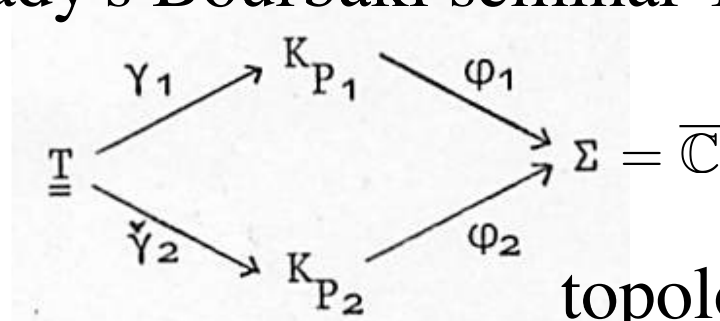
S. On a theorem of M. Rees for the matings of polynomials, in "The Mandelbrot set, Theme and Variations", Ed. Tan Lei, 2000

Matings of polynomials

Rational maps are more difficult than polynomials.
Try to understand the dynamics of a rational maps via a pair of polynomials P_1 and P_2 of degree $d > 1$



Douady's Bourbaki seminar 1982/1983

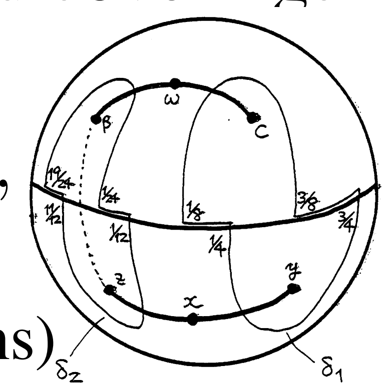


topological mating

Thurston's theorem, Thurston 1983, Douady-Hubbard1993
definition of Thurston equivalence for pcf branched coverings

Formal mating

$F = P_1 \coprod P_2 : S^2 \to S^2$, where $S^2 = \overline{\mathbb{D}}_1 \bigsqcup \overline{\mathbb{D}}_2 / \sim_{S^1_\infty}$,
$\overline{\mathbb{D}}_j = \mathbb{C} \cup S^1_\infty$ and $P_j : \overline{\mathbb{D}}_j \to \overline{\mathbb{D}}_j$ $(i = 1, 2)$
(modified formal mating after removing degenerate obstructions)



Q1. Is a (pcf) formal mating Thurston equivalent to a rational map?

Q2. If so, is it the same as the topological mating?

Thurston's theorem on the characterization of rational maps among self-branched covering of 2-sphere.

Let $f : S^2 \to S^2$ be a branched covering. (locally like $z \mapsto z^k$)

$Crit(f) = \{\text{critical pts of } f\}$, $P_f = \bigcup_{n=1}^{\infty} f^n(Crit(f))$ (post-crit. set)

Assume $\sharp P_f < \infty$. (*Post-critically finite*, PCF)

Two PCF branched coverings f and g are *equivalent*, $f \sim g$, if there exist two orientation preserving homeomorphisms $\theta_1, \theta_2 : S^2 \to S^2$ such that

$\theta_i(P_f) = P_g$ $(i = 1, 2)$, $\theta_1 = \theta_2$ on P_f, θ_1 and θ_2 are isotopic relative to P_f, and the following diagram commutes:

$$\begin{array}{ccc} S^2 & \xrightarrow{\theta_1} & S^2 \\ f \downarrow & & \downarrow g \\ S^2 & \xrightarrow{\theta_2} & S^2. \end{array}$$

Q. Given f as above, when is it equivalent to a rational map?

A multicurve Γ is a collection of disjoint simple closed curves in $S^2 \setminus P_f$ such that they are noy homotopic to a point or to a puncture, not homotopic to each other.

Define $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by $f_\Gamma : (m_\gamma)_{\gamma \in \Gamma} \mapsto (m'_\gamma)_{\gamma \in \Gamma}$ Thurston matrix

where $m'_\gamma = \sum_{\delta \in \Gamma} \sum_{\substack{\delta' \subset f^{-1}(\delta) \\ \delta' \sim \gamma}} \frac{m_\delta}{deg(f : \delta' \to \delta)}$.

$\lambda_\Gamma =$ leading eigenvalue of f_Γ (Thurston eigenvalue)

A multicurve Γ with $\lambda_\Gamma \geq 1$ is called Thurston obstruction.

Theorem (Thurston). (Published by Douady-Hubbard 1993)
A PCF branched covering $f : S^2 \to S^2$ (with $\sharp P_f \geq 5$) is equivalent to a rational map if and only if it has no Thurston obstruction, i.e., any invariant multicurve $\Gamma \subset S^2 \setminus P_f$ satisfies $\lambda_\Gamma < 1$.
Moreover when this condition holds, the equivalent rational map is unique. (Rigidity)

negative criterion: find one obstruction
positive criterion: check for all multicurves (cf. Dylan Thurston)

In this talk, we focus on the mateability question (Thurston eq to a rat map?)

Q1. Is a (pcf) formal mating Thurston equivalent to a rational map?

Theorem. (Rees & Tan Lei) For two pcf polynomials of degree two, they are mateable if and and only if they do not belong to the conjugate limbs of the Mandelbrot set.

A key ingredient is Levy cycle theorem.

A multicurve $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{p-1}\}$ is called a *Levy cycle* if each γ_i has an inverse image $\gamma'_{i-1} \subset f^{-1}(\gamma_i)$ which is homotopic to γ_{i-1} with $deg(f : \gamma'_{i-1} \to \gamma_i) = \pm 1$ ($i = 1, \ldots, p$ with $\gamma_p = \gamma_0$). By taking inverse images, Γ can be extended to be a Thurston obstruction.

Levy cycle theorem. (Levy, Rees) If f is a topological polynomial or a branched covering of degree 2, then f has a Thurston obstruction if and only if it has a Levy cycle.

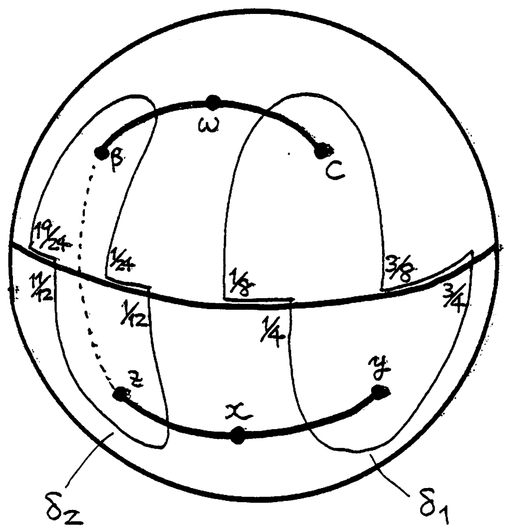
Levy cycle theorem allows us to derive a combinatorial condition from the topological (up to isotopy) one.

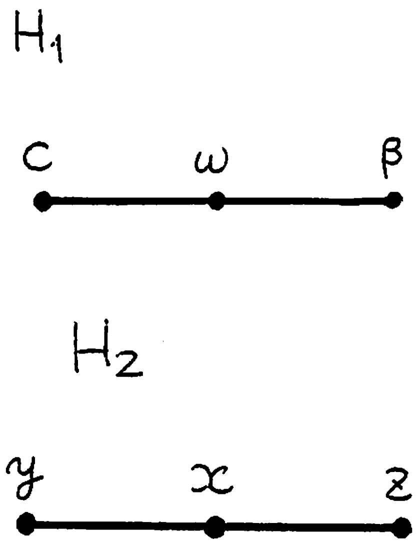
If f is "expanding", then the pull back of curves (together with the homotopy) will converge to a fixed object with some combinatorics.

$P_1 \coprod P_2$ is NOT "expanding", but if deg=1, the homotopy behaves nicely across the equator. (may assume tha homotopy is parallel to the equator)

Q. (Tan Lei 1988) Does the Levy cycle theorem hold for higher degree rational maps? Or for the formal matings?

NO: A cubic example (S. and Tan Lei)

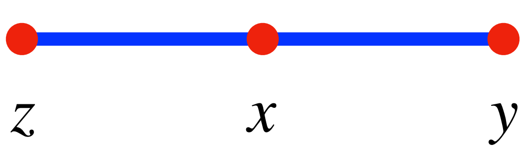


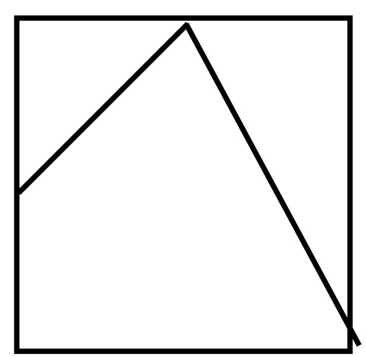


$$\omega = 0 \overset{3}{\to} c \to \beta \to \beta$$

$$x \overset{2}{\to} y \overset{2}{\to} z \to x$$

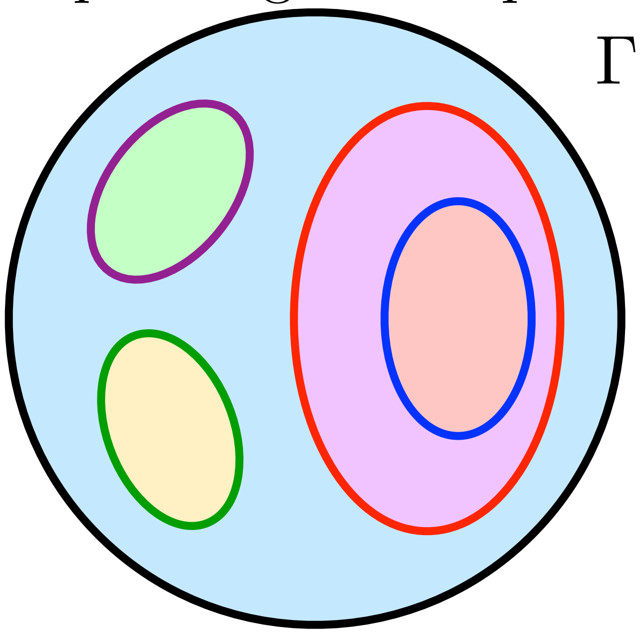
It was not first constructed as a mating, but from a tree of obstruction.



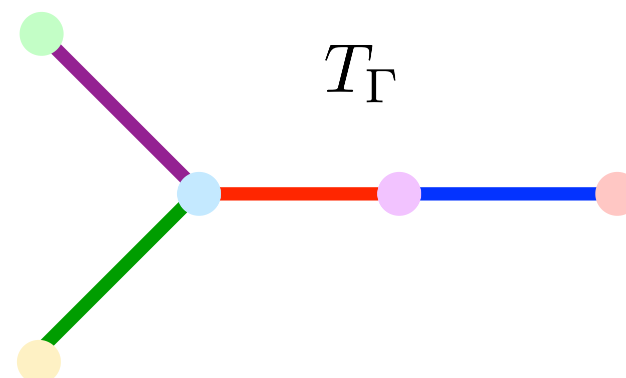


Tree associated to an invariant multicurve of a PCF map

Given a multicurve Γ on S^2, one can associate a tree $T = T_\Gamma$ so that each connected component of $S^2 \setminus \cup\Gamma$ corresponds to a vertex of T; each γ corresponds to an edge of T which connects the two vertices corresponding to components of $S^2 \setminus \cup\Gamma$ sharing γ as boundary.



Γ ("dual graph") cf. Work by Pilgrim



induced map $f : T_{f^{-1}(\Gamma)} \to T_\Gamma$ $\quad$ $T_\Gamma \overset{r}{\leftarrow} T_{f^{-1}(\Gamma)} \overset{f}{\rightarrow} T_\Gamma$

Assume that Γ is "irreducible in the sense of graph".

Take a positive eigenvector of the Thurston matrix for λ_Γ, one can define an eigenvector metric on the tree T_Γ.

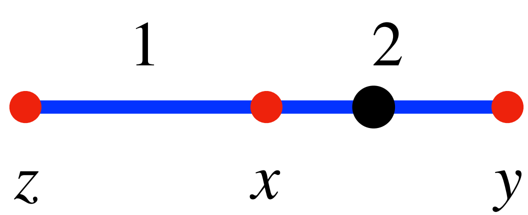
With respect to this metric, the induced map $f : T_\Gamma \to T_\Gamma$ is piecewise linear with local Lischitz const $= \lambda_\Gamma \times \deg(f : \delta' \to f(\delta'))$

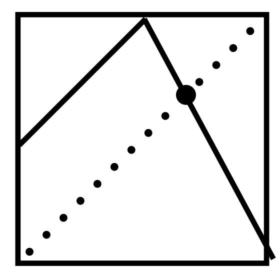
Mechanism for quadratic Levy cycle theorem

If f is a quadratic PCF map with a Thurston obstruction Γ, then Γ contains a Levy cycle.

Idea: Consider the tree map $f : T_\Gamma \to T_\Gamma$, which is piecewise expanding, because $\lambda_\Gamma \geq 1$. It has a fixed point, so look at the branches around the fixed point.
If there is a folding along a branch, there must be two critical points at the branching pt, so the degree on all the curves will be 1.
If no folding, then it must be isometric along periodic branches, this leads to a Levy cycle.

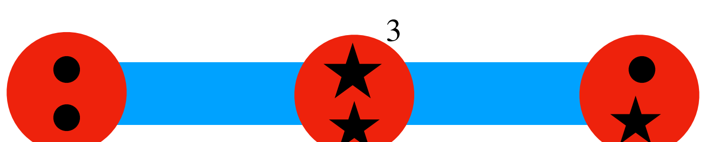
This argument fails as soon as there are 3 critical points.





$\lambda = 1$

Realize it as a branched cover



and later as a mating

Q. How can we detect the obstruction of a PCF branched covering maps? How about in the case of the matings of polynomials?

Partial answers: S.-Tan Lei, Tan Lei, Pilgrim-Tan Lei, etc.
difficulties in the infinite conditions in Thurston's theorem.
Positive criterion by Dylan Thurston.

Q. In the cubic example, is it a coincidence that the tree of obstruction looks similar to the Hubbard trees?

NO. We looked for the polynomials whose Hubbard tree is like T_Γ.

Q. How general is this phenomenon and how to use it in order to detect possible obstructions?

"In progress" since 1988: we want to say something like:

If two polynomials P_1 and P_2 are not mateable, then their Hubbard trees H_1 and H_2 have T_Γ as a “common factor”.

If two polynomials do not have a “common factor” with a certain property, then they are mateable.

We propose as a tentative definition of "factor" as follows:

Let $f_1 : T_1 \to T_1$ and $f_2 : T_2 \to T_2$ be two tree maps (piecewise linear maps). (H_2, f_2) is called a "factor" of (H_1, f_1), if there exists a forward invariant closed subset $X \subset T_1$ and a continuous sujective map $h : X \to T_2$ such that $h \circ f_1 = f_2 \circ h$ on X, and for each component J of $T_1 \setminus X$, h maps the boundary ∂J to a single point.

example: mating of Chebyshev map and real quadratic map

Related to:

Milnor-Thurston kneading theory on unimodal maps (allowable sequences)

Milnor: model for real cubic maps "stunted sawtooth map"

Thurston: core entropy, monotonicity along veins of *M*

"Claim". Let P_1 and P_2 be PCF polynomials of the same degree. Suppose that the formal mating $P_1 \coprod P_2$ has a Thurston obstruction Γ. Then (T_Γ, F_γ) is a "factor" of both (H_1, P_1) and (H_2, P_2), where H_j is the Hubbard tree of P_j. Moreover for external angles...

Possible application?

Fix P_2 in the cubic example, show that a polynomial $z \mapsto z^3 + c$ is mateable with P_2 if and only if they have the same type of obstruction as the cubic example.

$$x \xrightarrow{2} y \xrightarrow{2} z \rightarrow x$$

Restriction on the type of obstruction from the Hubbard tree H_j, for example, in terms of branch points, etc?

One can ask a question (apart from matings etc): Given a tree map what are the possible “factors”?

A Scenario for the construction of the "factor map" h

Let $f : S^2 \to S^2$ be a PCF map which contains a topological Hubbard tree H such that $P_{f|_H} \subset P_f$ (augmented) and all branch points of H are also in P_f. Assume that f has an obstruction Γ.

Take a representative of Γ with a minimal intersection, and take a refined Markov partition of H so that each segment intersect Γ at most once. This induces a first map $h_0 : H \to T_\Gamma$ such that if a segment J in H intersects $\gamma \in \Gamma$, then J is mapped to a segment corresp. to γ in T_Γ, otherwise a segment J will be mapped to a point in T_Γ (which is determined by the continuity).

This is far from canonical, because one can move the intersection by homotopy.

In general, $f^{-1}(\Gamma)$ has extra intersections, so fix an isotopy that erases them. (Which intersection should be erased is not unique.) After erasing the extra intersection, one can define $h_1 : H \to T_\Gamma$. Continuing recursively we obtain a sequence $h_n : H \to T_\Gamma$ which are related by $h_{n+1} = h_n \circ \eta_n$, where η is a monotone map on each segment on $H \setminus P_{f|_H}$.

Show that $\lim_{n\to\infty} h_n$ or $\lim_{n\to\infty} \eta_0 \circ \cdots \circ \eta_n$ exist and have desired properties.

Merci!