Two Moduli Spaces

John Milnor

Work with Araceli Bonifant

Stony Brook University

ANGERS, OCTOBER 2017

IN MEMORY OF TAN LEI

Outline: Two Examples.

The object of this talk will be to describe two examples of smooth group actions on smooth manifolds.

Easier Example (Divisors on $\mathbb{P}^1$):

> *The group $G(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$ of Möbius automorphisms of the Riemann sphere $\mathbb{P}^1$ acts on the space $\mathfrak{D}_n$ of effective divisors of degree n on $\mathbb{P}^1$, with quotient space $\mathfrak{D}_n/G(\mathbb{P}^1)$.*

Much Harder Example (Curves in $\mathbb{P}^2$):

> *The group $G(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$ of projective automorphisms of the complex projective plane $\mathbb{P}^2$, acts on the projective compactification $\mathfrak{C}_n$ of the space of algebraic curves of degree n in $\mathbb{P}^2$, with quotient space $\mathfrak{C}_n/G(\mathbb{P}^2)$.*

In both cases, some parts of the quotient space are beautiful objects to study, but other parts are rather nasty.

Basic Problem: Which parts are which?

A topological space Y is a T_1 ***-space*** if every point $\mathbf{p} \in Y$ is closed as a subset of Y.

Y is ***locally*** T_1 at $\mathbf{p}$ if some neighborhood of $\mathbf{p}$ is a T_1 -space.

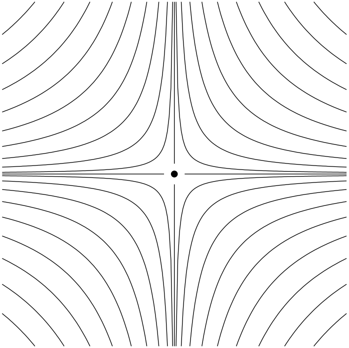
Easy Exercise. In any topological space Y, the subset consisting of all points at which Y is locally T_1 is itself a T_1 -space.

This subset can be described as the unique maximal open subset of Y which is a T_1 -space.

Locally Hausdorff Spaces

Definition. A space Y is ***locally Hausdorff*** at **p** if some neighborhood of **p** is a Hausdorff space.

Evidently the set of all points at which Y is locally Hausdorff is a well defined open subset of Y.



A Toy Example: The additive group G of real numbers acts on $\mathbb{R}^2$ by $\mathbf{g}_t(x, y) = (e^t x,\ e^{-t} y)$. The quotient space $\mathbb{R}^2/G$ is locally T_1 and even locally Hausdorff, **except at** $(0, 0)$.
The quotient $\big(\mathbb{R}^2 \setminus \{(0, 0)\}\big)/G$ is locally Hausdorff everywhere,
but is not Hausdorff.

Part 2. The Space $\mathfrak{D}_n$ of Degree n Divisors on $\mathbb{P}^1$.

Definition: An ***effective divisor*** $\mathcal{D}$ of degree n on the Riemann sphere $\mathbb{P}^1$ is a formal sum

$$\mathcal{D} \;=\; m_1\langle z_1\rangle + \cdots + m_k\langle z_k\rangle \;,$$

where the $m_j > 0$ are integers with $\sum_j m_j = n$, and the z_j are distinct points of $\mathbb{P}^1$.

Each such $\mathcal{D}$ can be identified with the set of zeros, counted with multiplicity, for some non-zero homogeneous polynomial

$$\Phi(x, y) \;=\; c_0 x^n \,+\, c_1 x^{n-1} y \,+\, \cdots \,+\, c_n y^n \;.$$

It follows that the space $\mathfrak{D}_n$ of all such divisors is isomorphic to the projective space $\mathbb{P}^n$.

The group $G = G(\mathbb{P}^1)$ of Möbius automorphisms of $\mathbb{P}^1$ acts on $\mathfrak{D}_n$.

Two integer invariants under the action of G:

- *The number of points $k = \#|\mathcal{D}|$ in the* ***support***

$$|\mathcal{D}| = \{z_1, \ldots, z_k\} \subset \mathbb{P}^1 \;.$$

- *The maximum $m_{\mathbf{max}} = \mathbf{max}\{m_1, \ldots, m_k\}$ of the multiplicities of the various points of $|\mathcal{D}|$.*

Finite Stabilizer $\iff \quad \#|\mathcal{D}| \geq 3$.

Definition. The ***stabilizer*** $G_{\mathcal{D}}$ of a divisor $\mathcal{D}$ is the subgroup of G consisting of all $\mathbf{g} \in G$ with $\mathbf{g}(\mathcal{D}) = \mathcal{D}$.

Lemma. *The stabilizer $G_{\mathcal{D}}$ is finite if and only if the support $|\mathcal{D}|$ contains at least three elements.*

Proof. For any $\mathcal{D}$ there is a natural homomorphism $G_{\mathcal{D}} \to \mathcal{S}_{|\mathcal{D}|}$, where $\mathcal{S}_{|\mathcal{D}|}$ is the symmetric group consisting of all permutations of the finite set $|\mathcal{D}|$. If $\#|\mathcal{D}| \geq 3$, since any Möbius transformation which fixed three distinct points must be the identity, it follows that $G_{\mathcal{D}}$ is finite, isomorphic to a subgroup of $\mathcal{S}_{|\mathcal{D}|}$.

Now suppose that $\#|\mathcal{D}| \leq 2$. After a Möbius transformation, we may assume that $|\mathcal{D}| \subset \{0, \infty\}$. (Here I am identifying the Riemann sphere with $\mathbb{C} \cup \{\infty\}$.) The group $G_{\mathcal{D}}$ then contains infinitely many transformations of the form

$$\mathbf{g}_{\kappa}(z) = \kappa z \quad \text{with} \quad \kappa \neq 0 . \quad \square$$

The Moduli Space for Divisors.

Let $\mathfrak{D}_n^{\text{fstab}}$ be the open subset of $\mathfrak{D}_n$ consisting of all divisors with finite stabilizer (or all divisors with $\#|\mathcal{D}| \ge 3$).

Proposition 1. *Every G-orbit*

$$((\mathcal{D})) = \{\mathbf{g}(\mathcal{D})\ ;\ \mathbf{g} \in G\}$$

in $\mathfrak{D}_n^{\text{fstab}}$ is closed as a subset of $\mathfrak{D}_n^{\text{fstab}}$.

In other words, every divisor $\mathcal{D}'$ which belongs to the topological boundary $\overline{((\mathcal{D}))} \smallsetminus ((\mathcal{D}))$ must have infinite stabilizer.

Definition. This quotient $\mathfrak{M}_n = \mathfrak{D}_n^{\text{fstab}}/G$ will be called the ***moduli space*** for divisors, under the action of G.

Thus Proposition 1 implies that $\mathfrak{M}_n$ is a T_1-space.

To prove Proposition 1, we must study elements of G which are "close to infinity" in G.

Distortion Lemma for Möbius Transformations.

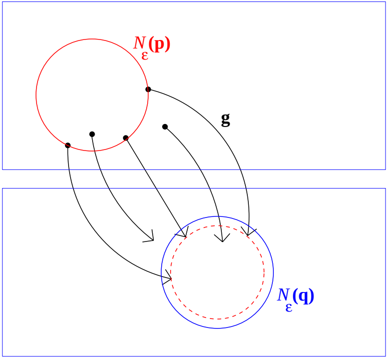
Using the spherical metric on $\mathbb{P}^1$, let $N_\varepsilon(\mathbf{p})$ be the open ε-neighborhood of $\mathbf{p}$.

Lemma. *For any* $\varepsilon > 0$ *there is a compact set*

$$K = K_\varepsilon \subset G$$

with the following property: For any $\mathbf{g} \notin K$*, there are (not necessarily distinct) points* $\mathbf{p}$ *and* $\mathbf{q}$ *such that*

$$\mathbf{g}(N_\varepsilon(\mathbf{p})) \cup N_\varepsilon(\mathbf{q}) = \mathbb{P}^1 .$$



Thus points outside of $N\varepsilon(\mathbf{p})$ map inside $N_\varepsilon(\mathbf{q})$.

(**Proof Outline.** The proof for the group of diagonal transformations $\mathbf{d}(x:y) = (\kappa\, x : y)$ is easy. But any $\mathbf{g} \in G$ can be written as a product $\mathbf{g} = \mathbf{r} \circ \mathbf{d} \circ \mathbf{r}'$ where $\mathbf{r}$ and $\mathbf{r}'$ are rotations of the Riemann sphere and $\mathbf{d}$ is diagonal. . . .)

Proof of Proposition 1.

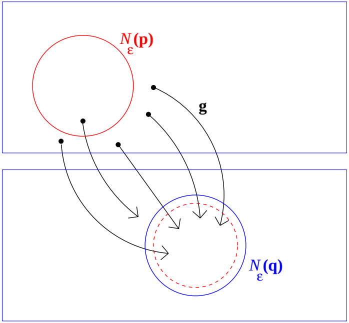
To prove: *Every G-orbit* $((\mathcal{D})) \subset \mathfrak{D}_n^{\text{fstab}}$ *is closed as a subset of* $\mathfrak{D}_n^{\text{fstab}}$ *.*

Choose ε small enough so that any two points of $|\mathcal{D}|$ have distance $> 2\varepsilon$ from each other.

$\Longrightarrow$ No ε-ball contains more than one point of $|\mathcal{D}|$.

Given any $\mathbf{g} \notin K_\varepsilon$, choose $\mathbf{p}$ and $\mathbf{q}$ as in the Distortion Lemma. It follows that:

all but possibly one of the points of $\mathbf{g}\big(|\mathcal{D}|\big)$ lie in $N_\varepsilon(\mathbf{q})$.



Now suppose that we are given a sequence of group elements $\mathbf{g}_j \notin K_{\varepsilon_j}$ with $\varepsilon_j \to 0$, and suppose that the sequence of divisors

$$\mathbf{g}_j(\mathcal{D}) \in ((\mathcal{D}))$$

converges to some $\mathcal{D}'$.

Then $\#|\mathcal{D}'| \le 2$,

hence $\mathcal{D}' \notin \mathfrak{D}_n^{\text{fstab}}$. $\square$

$\mathfrak{M}_n$ is Hausdorff only for Small n.

Let $\pi : \mathfrak{D}_n^{\text{fstab}} \to \mathfrak{M}_n$ be the projection map.

Proposition 2. *The moduli space*

$$\mathfrak{M}_n = \mathfrak{D}_n^{\text{fstab}}/G$$

is a Hausdorff space only if $n \le 4$.

For any n*, the open set consisting of points* $\pi(\mathcal{D})$ *with* $m_{\textbf{max}} < n/2$ *is a Hausdorff space.*

However, if $n \ge 5$*, then points* $\pi(\mathcal{D})$ *with* $m_{\textbf{max}} \ge n/2$ *are not even locally Hausdorff.*

Low degree examples:

$\mathfrak{M}_3$ is a point.

$\mathfrak{M}_4 \cong \mathbb{P}^1$. [Proof Outline: Four distinct points in $\mathbb{P}^1$ determine a 2-fold branched covering which is an elliptic curve; characterized by the classical invariant $j(\mathcal{C}) \in \mathbb{C}$. But there is one other G-orbit $((\mathcal{D})) \subset \mathfrak{D}_4^{\text{fstab}}$ consisting of divisors with $\#|\mathcal{D}| = 3$. Therefore: $\mathfrak{M}_4 \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$.]

A Non Locally Hausdorff Example.

Choose two divisors

$$\mathcal{D} = \mathcal{D}_h + h\langle\infty\rangle \qquad \text{and} \qquad \mathcal{D}' = \mathcal{D}'_h + h\langle\infty\rangle \ ,$$

of degree $n = 2h \geq 6$, ***which are not in the same*** ***G-orbit,*** with both $|\mathcal{D}_h|$ and $|\mathcal{D}'_h|$ contained in $\mathbb{C}$.
Note that $m_{\max} = h = n/2$ for both.

Let $\mathbf{g}_r(z) = r^2/z$, with $r \gg 1$;

so that $|z| < r \iff |\mathbf{g}_r(z)| > r$.

Then the two divisors $\mathcal{D}_h + \mathbf{g}_r(\mathcal{D}'_h)$ and $\mathcal{D}'_h + \mathbf{g}_r(\mathcal{D}_h)$ belong to the same G-orbit.

As $r \to \infty$, the first converges to $\mathcal{D}$
and the second converges to $\mathcal{D}'$.

Thus every neighborhood of $\pi(\mathcal{D}) \in \mathfrak{M}_n$
intersects every neighborhood of $\pi(\mathcal{D}')$.

Since $\mathcal{D}'$ can be arbitrarily close to $\mathcal{D}$, this proves that $\mathfrak{M}_{2h}$ is not locally Hausdorff at the point $\pi(\mathcal{D})$.

Proper and Locally Proper Actions.

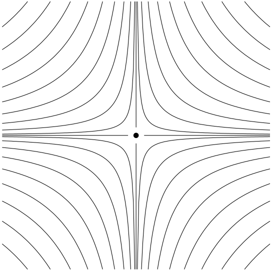
Let G be a Lie group G acting on a locally compact space X.

Definition. The action is ***proper at a pair*** $(\mathbf{x}, \mathbf{y}) \in X \times X$ if, for some neighborhood $U \times V$ of $(\mathbf{x}, \mathbf{y})$, there exists a compact set $K \subset G$ such that:

Every $\mathbf{g}$ which satisfies $\mathbf{g}(U) \cap V \neq \emptyset$ belongs to K.

If this is true for all $(\mathbf{x}, \mathbf{y})$, then the action is ***proper***.

If it is true throughout a neighborhood of $\mathbf{x}$ then the action is ***locally proper*** at $\mathbf{x}$.



In our toy example, the action is not proper; but it is locally proper away from the origin.

Exercise: Proper action $\Rightarrow$ X/G Hausdorff;

and locally proper action $\Rightarrow$ X/G locally Hausdorff.

To Prove: $m_{\mathbf{max}} < n/2$ implies Hausdorff.

Let $\mathcal{D},\ \mathcal{D}' \in \mathfrak{D}_n$ be two divisors, both satisfying the condition that $m_{\mathbf{max}} < n/2$.

Lemma. *We can choose neighborhoods $\mathfrak{N}_{\mathcal{D}}$ of $\mathcal{D}$ and $\mathfrak{N}_{\mathcal{D}'}$ of $\mathcal{D}'$ in $\mathfrak{D}_n$, and a compact set $K \subset G$, such that any group element satisfying*

$$\mathbf{g}(\mathfrak{N}_{\mathcal{D}}) \cap \mathfrak{N}_{\mathcal{D}'} \neq \emptyset$$

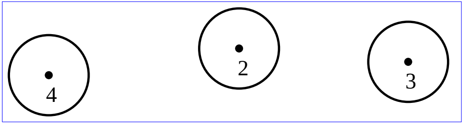
must belong to K.

In other words, the action of G is proper throughout the G-invariant set where $m_{\mathbf{max}} < n/2$. Since proper action implies a Hausdorff quotient, this Lemma implies that the corresponding open subset of $\mathfrak{M}_n$ is a Hausdorff space.

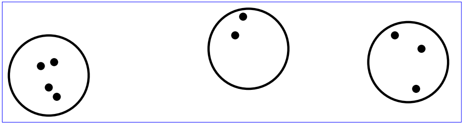
First step of proof: Choose $\varepsilon > 0$ small enough so that so that any two points of $|\mathcal{D}|$ or of $|\mathcal{D}'|$ have distance $> 4\,\varepsilon$.

Proof (continued): Next define $\mathfrak{N}_\varepsilon(\mathcal{D})$.

ε-balls around the points of $|\mathcal{D}|$:



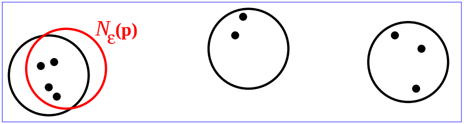
$\widehat{\mathcal{D}} \in \mathfrak{N}_\varepsilon(\mathcal{D})$:



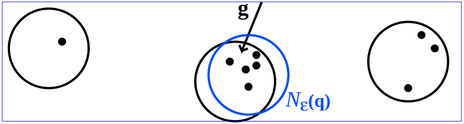
Now suppose that $\mathbf{g}(\widehat{\mathcal{D}}) \in \mathfrak{N}_\varepsilon(\mathcal{D}')$, with $\mathbf{g} \notin K_\varepsilon$.

Then there are $\mathbf{p}$ and $\mathbf{q}$ with $\mathbf{g}\big(\mathbb{P}^2 \smallsetminus N_\varepsilon(\mathbf{p})\big) \subset N_\varepsilon(\mathbf{q})$.

Here $N_\varepsilon(\mathbf{p})$ contains $< n/2$ points of $\widehat{\mathcal{D}}$,



hence $N_\varepsilon(\mathbf{q})$ contains $> n/2$ points of $\mathbf{g}(\widehat{\mathcal{D}})$;



which contradicts the hypothesis. □

Part 3. Curves in the Projective Plane.

Definition. An ***effective 1-cycle*** of degree $n \geq 1$ on the complex projective plane $\mathbb{P}^2$ is a formal sum

$$\mathcal{C} = m_1 \cdot \mathcal{C}_1 + \cdots + m_k \cdot \mathcal{C}_k ,$$

where each $\mathcal{C}_j$ is an irreducible complex curve, where the $m_j \geq 1$ are integers, and where $n = \sum_j m_j \deg(\mathcal{C}_j)$.

The space $\mathfrak{C}_n$ of all effective 1-cycles can be given the structure of a complex projective space of dimension $n(n+3)/2$. (In fact each non-zero homogeneous polynomial $\Phi(x, y, z)$ of degree n has a zero locus consisting of irreducible curves $\mathcal{C}_j$, each counted with some multiplicity $m_j \geq 1$; yielding a 1-cycle.)

The group $G = G(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$ of all automorphisms of $\mathbb{P}^2$ acts on $\mathbb{P}^2$ and hence on the space $\mathfrak{C}_n$.

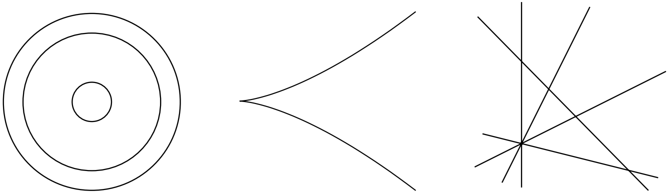
The stabilizer $G_{\mathcal{C}}$ of $\mathcal{C} \in \mathfrak{C}_n$ is just the group consisting of all projective automorphisms which map $\mathcal{C}$ to itself.

This stabilizer $G_{\mathcal{C}}$ may be either finite or infinite.

W-curves (and cycles).

Curves with infinite stabilizer were first studied by Felix Klein and Sophus Lie, who called them ***W-curves***.

Some examples:



Let $\mathfrak{W}_n \subset \mathfrak{C}_n$ be the algebraic set consisting of all cycles with infinite stabilizer. ($\mathfrak{W}_n$ is a union of finitely many maximal irreducible subvarieties of $\mathfrak{C}_n$, of varying dimension.)

Note: $\mathcal{C}$ has finite stabilizer if and only if the G-orbit $(\!(\mathcal{C})\!) \subset \mathfrak{C}_n$ has dimension 8.

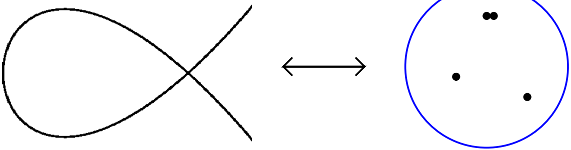
In fact $\dim (\!(\mathcal{C})\!) + \dim(G_{\mathcal{C}}) = \dim(G) = 8$,
where $\dim(G_{\mathcal{C}}) = 0 \iff G_{\mathcal{C}}$ is finite.

The Moduli Space $\mathbb{M}_n$.

The complement $\mathfrak{C}_n^{\text{fstab}} = \mathfrak{C}_n \smallsetminus \mathfrak{W}_n$ is the open set consisting of all cycles with ***finite stabilizer.***

Definition. The quotient space $\mathbb{M}_n = \mathfrak{C}_n^{\text{fstab}}/G$, will be called the ***moduli space*** for plane cycles of degree n.

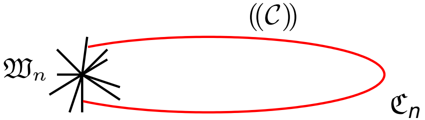
Examples. $\mathbb{M}_1 = \mathbb{M}_2 = \emptyset$.
The moduli space $\mathbb{M}_3$ for cubic curves in $\mathbb{P}^2$ is canonically isomorphic to the moduli space $\mathfrak{M}_4$ for divisors in $\mathbb{P}^1$.
Each has two "ramified points" corresponding to points with extra symmetry (= larger stabilizer). Each also has one "improper point" where the group action is not proper.



Thus $\mathbb{M}_3 \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$.

$\mathbb{M}_n$ is a T_1-space.

Cartoon of $\mathfrak{C}_n$, showing a typical G-orbit in red:



The topological boundary of any G-orbit in $\mathfrak{C}_n$ is contained in the closed subset $\mathfrak{W}_n$.

[Ghizzetti 1936; Aluffi and Faber 2010.]

$\Longrightarrow$ Every G-orbit of cycles with finite stabilizer is closed as a subset of $\mathfrak{C}_n^{\text{fstab}}$.

$\Longrightarrow$ Every point in $\mathbb{M}_n$ is a closed set.

Virtual Flex Points

Let $\mathfrak{U}_n \subset \mathfrak{C}_n$ be the open subset consisting of all $\mathcal{C} \in \mathfrak{C}_n$ which satisfy the following two conditions:

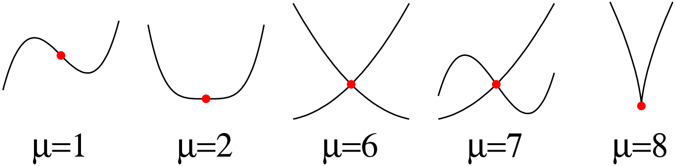
(1) $\mathcal{C}$ contains no line.

(2) $\mathcal{C}$ contains no component with multiplicity ≥ 2.

Definition. For $\mathcal{C} \in \mathfrak{U}_n$, a point $\mathbf{p} \in |\mathcal{C}|$ will be called a ***virtual flex point*** (or ***VFP***) if it is either a flex point or a singular point. Each VFP has a ***multiplicity*** $\mu(\mathbf{p}) \geq 1$ satisfying the following:

Under a generic small perturbation of the curve, $\mathbf{p}$ *will split into* $\mu(\mathbf{p})$ *nearby flex points.*

Some Examples:



A Criterion for Proper Action.

Fixing some curve $\mathcal{C} \in \mathfrak{U}_n$, for any set $S \subset \mathbb{P}^2$, define

$$\mu(S) = \mu_{\mathcal{C}}(S) \;=\; \sum_{\mathbf{p} \in S \cap \mathcal{C}} \mu(\mathbf{p}) \;.$$

The total number of VFP is $\mu(\mathcal{C}) = 3n(n-2)$.

We can also introduce the probability measure

$$\widehat{\mu}(S) \;=\; \mu(S)/\mu(\mathcal{C}) \;\in\; [0, 1] \;.$$

Now define two invariants $\mathbf{pmax}(\mathcal{C}) \le \mathbf{Lmax}(\mathcal{C}) \le 1$:

pmax$(\mathcal{C})$ is the maximum of $\widehat{\mu}(\mathbf{p})$ over all points $\mathbf{p} \in \mathcal{C}$.

Lmax$(\mathcal{C})$ is the maximum of $\widehat{\mu}(L)$ over all lines $L \subset \mathbb{P}^2$.

Theorem (Preliminary Version).

If $\mathbf{pmax}(\mathcal{C}) + \mathbf{Lmax}(\mathcal{C}) < 1$,

then the action of G is locally proper at $\mathcal{C}$.

The square of pairs (**pmax**, **Lmax**)

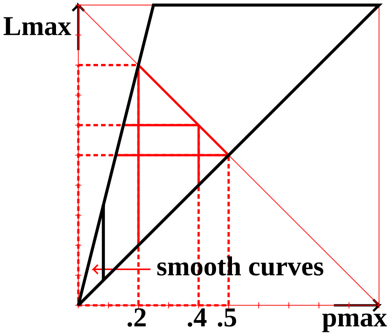


Illustration for degree $n = 4$, showing three typical rectangles where the action is known to be proper.

Here is a more precise statement:
Given any constant $0 < \kappa \le 1/2$, let $\mathfrak{U}_n(\kappa)$ be the set of all $\mathcal{C} \in \mathfrak{U}_n$ such that **pmax**$(\mathcal{C}) < \kappa$ and **Lmax**$(\mathcal{C}) < 1 - \kappa$.

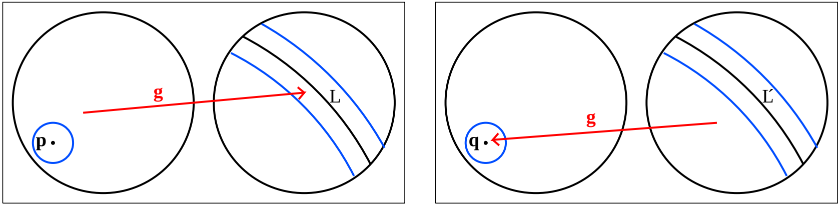
Theorem: The action of G on each $\mathfrak{U}_n(\kappa)$ is proper.

Corollary: The action of G on the space of smooth degree n curves is proper.

The Distortion Lemma for $\mathbb{P}^2$.

Lemma. *Given $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset G(\mathbb{P}^2)$ with the following property.*

For any $\mathbf{g} \notin K_\varepsilon$ there exists either:



(1) a point $\mathbf{p} \in \mathbb{P}^2$ and a line $L \subset \mathbb{P}^2$ such that

$$\mathbf{g}\big(N_\varepsilon(\mathbf{p})\big) \cup N_\varepsilon(L) = \mathbb{P}^2$$

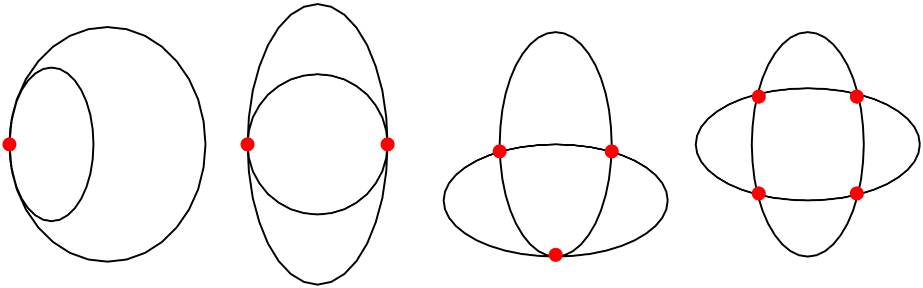
(so that $\mathbf{g}$ maps every point outside of $N_\varepsilon(\mathbf{p})$ into $N_\varepsilon(L)$),

or **(2)** a line $L' \subset \mathbb{P}^2$ and a point $\mathbf{q} \in \mathbb{P}^2$ such that

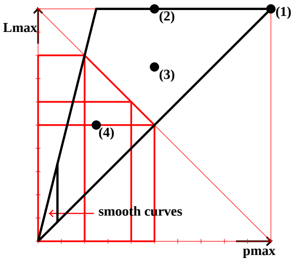
$$\mathbf{g}\big(N_\varepsilon(L)\big) \cup N_\varepsilon(\mathbf{q}) = \mathbb{P}^2$$

(so that $\mathbf{g}$ maps every point outside of $N_\varepsilon(L')$ into $N_\varepsilon(\mathbf{q})$).

Four examples, showing **pmax** + **Lmax**



$1 + 1 = 2$ $.5 + 1 = 1.5$ $.5 + .75 = 1.25$ $.25 + .5 = .75$



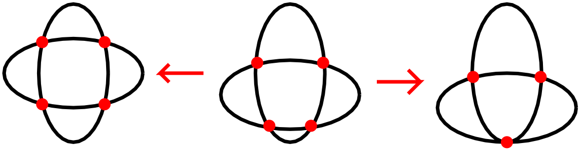
The first two are W-curves.

Only the right hand example has locally proper action.

An Improper Example.

Lemma. In a region where the action is proper,
the function $\mathcal{C} \mapsto G_{\mathcal{C}}$ is ***upper semicontinuous***:

If $\mathcal{C}_j \to \mathcal{C}$ then $\limsup_j(G_{\mathcal{C}_j}) \subset G_{\mathcal{C}}$.



$|G_{\mathcal{C}}| = 8$ $\qquad |G_{\mathcal{C}_j}| = 4$ $\qquad |G_{\mathcal{C}}| = 2$

The action is proper near the two left hand curves,
but not near the right hand curve.

Unknown:

Is the moduli space $\mathbb{M}_4$ locally Hausdorff
near the image of the right hand curve ?

References.

- P. Aluffi and C. Faber, *Limits of PGL(3)-translates of plane curves,* J. Pure Appl. Algebra **214** (2010) 526–547 and 548–564.
- A. Bonifant and J. Milnor, *On Real and Complex Cubic Curves,* arXiv:1603.09018v2 [math.AG]. To appear, L'Enseign. Math. **63**.
- A. Bonifant and J. Milnor, *Improper Group Actions, Orbifolds, and Plane Curves,* Manuscript in Preparation.
- A. Ghizzetti, *Determinazione delle curve limiti di un sistema continuo* ∞^1 *di curve piane omografiche,* Rendiconti Reale accademia dei Lincei. Classe di scienze fisiche, matematiche e naturali, **23** (1936) 261–264.
- F. Klein and S. Lie, *Ueber diejenigen ebenen Curven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen,* Math. Ann. **4** (1871), 50–84.
- D. Mumford et al., "Geometric Invariant Theory", Chapter 4, Section 2, Springer-Verlag-Berlin 1965, 1994.