ON COMBINATORIAL TYPES OF CYCLES UNDER THE MULTIPLICATION BY k MAP OF THE CIRCLE.

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Université d'Angers October 25, 2017

Notation and goal.

- Let $\mathbf{m}_k : \mathbb{T} \to \mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the multiplication by $k \geq 2$ map of the circle

$$\mathbf{m}_k(x) = kx \pmod{\mathbb{Z}}.$$

- The central question of this work is whether a given combinatoric $\sigma \in \mathcal{C}_q$ and or combinatorial type τ in $\mathcal{C}_q$ has a realization under $\mathbf{m}_k$ and if it does, how many such realizations there are.

Motivation I

There is a natural way to associate to each q-periodic point z for $\mathbf{m}_k$ belonging to a q-cycle $0 < z_1 < \ldots < z_q < 1$, say $z = z_j$, a pair of q-periodic points (x_j, y_j) characterized as follows :

- $y_j - x_j = \dfrac{(k+1)^{q-1}}{((k+1)^q - 1)}$,
- Their cycles are interlaced

$$0 < x_1 < y_1 < x_2 < y_2 < \ldots x_q < y_q < 1$$

- There is a monotone projection $P : \mathbb{T} \longrightarrow \mathbb{T}$ with $P(0) = 0$, $P([x_j, y_j]) = z_j$ and semi-conjugating $\mathbf{m}_{k+1}$ to $\mathbf{m}_k$ on $\mathbb{T} \setminus]x_j, y_j[$.

Connectedness locus for $\lambda z^2 + z^3$

e.g $z = \frac{3}{5}$ with $\mathbf{m}_2$-orbit

$$\left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\}$$

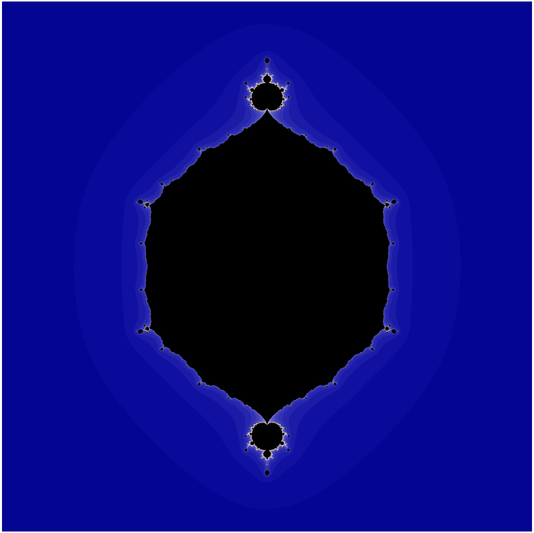
gives $(x_3, y_3) = (\frac{29}{80}, \frac{56}{80})$

with $\mathbf{m}_3$ orbits

$$\left\{ \frac{7}{80}, \frac{21}{80}, \frac{29}{80}, \frac{63}{80} \right\}$$

and

$$\left\{ \frac{8}{80}, \frac{24}{80}, \frac{56}{80}, \frac{72}{80} \right\}.$$



Motivation II

- I view the above as saying that for every periodic point z for $\mathbf{m}_k$ there is a pair of neighbouring periodic orbits for $\mathbf{m}_{k+1}$ with the same combinatorics and with critical interval corresponding to z.
- This motivates the following questions :
- Which combinatorics exists for $\mathbf{m}_{k+1}$, but does not exist for $\mathbf{m}_k$?
- How does the number of orbits with a given combinatorics grow with the degree k ?
- For rotation orbits with rational rotation number the answers to these questions are known.
- In fact for each irreducible rotation number p/q, $\mathbf{m}_2$ has a unique such orbit and Goldberg showed that in the general case, the number of such orbits is given by

$$\binom{q+k-2}{q}$$

Cyclic Permutations

- We shall use cyclic permutations to represent combinatorics of periodic orbits on the circle $\mathbb{T}$.
- Denote by $\mathcal{S}_q$ the group of permutations of q symbols, which we take to be the representatives $\{1, \ldots, q\}$ of the cyclically ordered set $\mathbb{Z}/q\mathbb{Z}$.
- Denote by $\mathcal{C}_q \subset \mathcal{S}_q$ the set of q-cycles σ in $\mathcal{S}_q$:

$$\sigma = (1\ \sigma(1)\ \sigma^2(1)\ \ldots\ \sigma^{q-}(1))$$

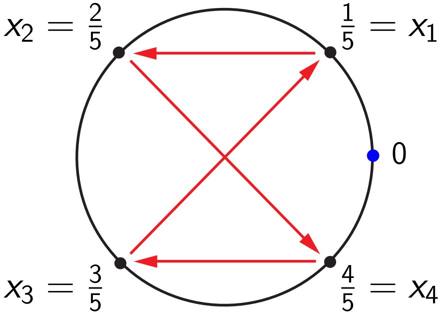
- And denote by $\mathcal{R}_q \subset \mathcal{S}_q$ the rotation group, that is the group generated by the q-cycle

$$\rho = (1\ 2\ \ldots\ q)$$

with rotation number $1/q$.

What is a combinatorics ? I

- Consider again the "Cocapeli"-orbit $\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$ under $\mathbf{m}_2$.



- We can view this as the representation $x_i \mapsto x_{\sigma(i)}$ of the cyclic permutation $\sigma = (1\ 2\ 4\ 3)$ acting on the set $\{1, 2, 3, 4\}$ representing the cyclically ordered set $\mathbb{Z}/4\mathbb{Z}$.
- Note that $0 \in \mathbb{T}$ IS NOT $0 \equiv 4 \in \mathbb{Z}/4\mathbb{Z}$.

What is a combinatorics ? II

- We shall use $\sigma = (1\ 2\ 4\ 3)$ as a synonym for the combinatorics of the orbit $\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$ under $\mathbf{m}_2$.
- More generally if $0 < x_1 < x_2 < \ldots < x_q < 1$ and

$$f : \{x_1, \ldots, x_q\} \longrightarrow \{x_1, \ldots, x_q\}$$

is a cyclic dynamics we shall say that the orbit $\{x_1, \ldots, x_q\}$ has *combinatorics* $\sigma \in \mathcal{C}_q$ iff

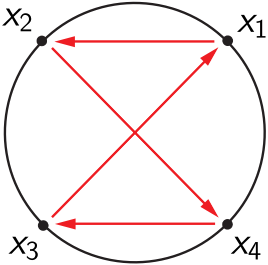
$$\forall i : f(x_i) = x_{\sigma(i)}.$$

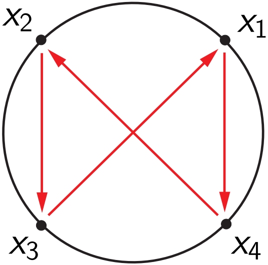
And we shall call any $\sigma \in \mathcal{C}_q$ a q-combinatorics.

A few numbers

- For each q the number of q-combinatorics is $(q-1)!$.
- For each $k \geq 2$ and q there are at most $\dfrac{k^q}{q}$ periodic orbits for $\mathbf{m}_k$ of period q.
- So for each fixed k and sufficiently large q the majority of the q-combinatorics are not realized by $\mathbf{m}_k$.
- The next slide shows as examples the four possible non-rotational 4-combinatorics

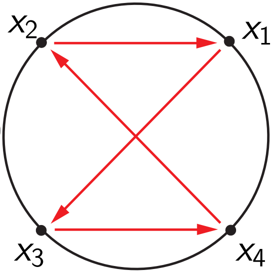
$\sigma_1 = (1\ 2\ 4\ 3)$

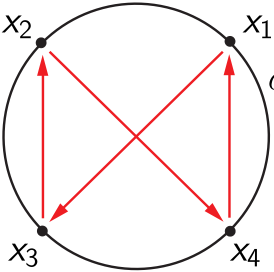




$\sigma_2 = (1\ 4\ 2\ 3)$

$\sigma_3 = (1\ 3\ 4\ 2)$





$\sigma_4 = (1\ 3\ 2\ 4)$

Botanics of combinatorics

- Only σ_1 is realized by $\mathbf{m}_2$, uniquely by our "Cocapeli"-orbit $\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$.
- The others however are each uniquely realized by $\mathbf{m}_3$:

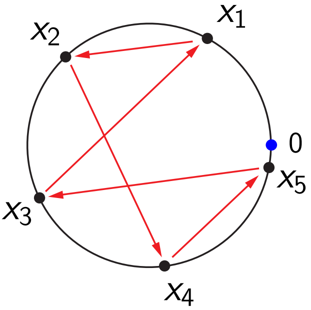
$$\sigma_2 = (1\ 4\ 2\ 3) : \qquad \left\{\frac{23}{80}, \frac{47}{80}, \frac{61}{80}, \frac{69}{80}\right\}$$

$$\sigma_3 = (1\ 3\ 4\ 2) : \qquad \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}$$

$$\sigma_4 = (1\ 3\ 2\ 4) : \qquad \left\{\frac{11}{80}, \frac{19}{80}, \frac{33}{80}, \frac{57}{80}\right\}$$

A 5-cycle example

$$\sigma = (1\ 2\ 4\ 5\ 3)$$



- This combinatorics is not realized by $\mathbf{m}_2$ either.
- It is however uniquely realized by $\mathbf{m}_3$:

$$\sigma = (1\ 2\ 4\ 5\ 3) : \qquad \left\{ \frac{8}{121}, \frac{24}{121}, \frac{43}{121}, \frac{72}{121}, \frac{95}{121} \right\}$$

Intervals in $\mathbb{Z}/q\mathbb{Z}$ and "lengths"

Definition

For $1 \leq i, j \leq q$ define the closed interval $[i, j]$ in $\mathbb{Z}/q\mathbb{Z}$ as :

$$[i,j] = \begin{cases} \{i, i+1, \ldots, j\} & \text{if } i < j, \\ \{i, (i+1), \ldots, (j+q)\} & \text{if } j < i. \end{cases}$$

And the length $||[i,j]|| := \#[i,j] - 1$ so that

$$||[i,j]|| = j - i \quad \text{if} \quad i \leq j \quad \text{and} \quad ||[i,j]|| = j + q - i \quad \text{if} \quad j < i.$$

All subsets of $\mathbb{Z}/q\mathbb{Z}$ are closed but we shall use the notion $[i, j)$ to indicate the "open interval $[i, j]$ minus the right end point

The degree of a cycle.

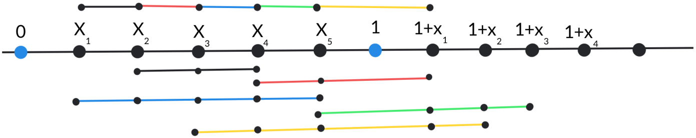
Definition

For $\sigma \in \mathcal{C}_q$ define $\deg(\sigma)$ as the integer :

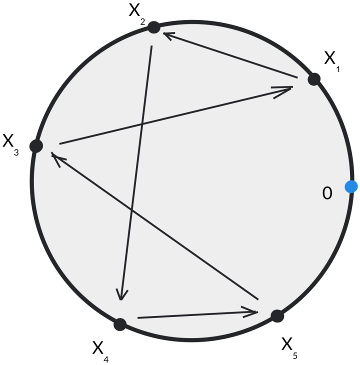
$$\deg(\sigma) = \frac{1}{q} \sum_{j=1}^{q} |[\sigma(j), \sigma(j+1)]|$$

- The degree of σ is equal to the descent number $\text{des}(\sigma)$ of the permutation σ as defined in combinatorial analysis.
- $\deg(1243) = \deg(1423) = \deg(1342) = \deg(1324) = 2$
- $\deg(12453) = 3$
- $\deg(\sigma) = 1$ if and only if σ is a rotation cycle.

Example $\sigma = (1\ 2\ 4\ 5\ 3)$



(1 2 4 5 3)



Topological realization

Definition

A *(topological) realization* of the cycle $\sigma \in \mathcal{C}_q$ is a pair $(f, \mathcal{O})$, where $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a positively oriented covering map, $\mathcal{O} = \{x_1, \ldots, x_q\}$, $0 < x_1 < \ldots, x_q < 1$ is a period q orbit of f, and $f(x_i) = x_{\sigma(i)}$ for all i.

The *degree* of the realization $(f, \mathcal{O})$ is the mapping degree of f.

A realization of σ is *minimal* if it has the smallest possible degree among all realizations.

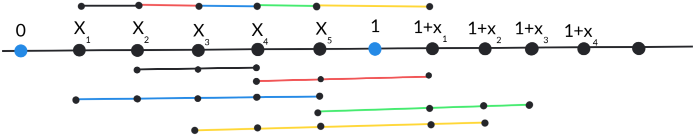
- For any $x \neq y \in \mathbb{T}$ let $[x, y]$ denote the closed interval in $\mathbb{T}$ with end points x, y such that for any z in the corresponding open interval $]x, y[$ the triple (x, z, y) is positively oriented.
- Equivalently let $\Pi : \mathbb{R} \longrightarrow \mathbb{T}$ denote the natural projection. Then $[x, y] = \Pi([\widehat{x}, \widehat{y}])$, where $\Pi(\widehat{x}) = x$ and $\Pi(\widehat{y}) = y$ and $\widehat{x} < \widehat{y} < \widehat{x} + 1$.

- For $(f, \mathcal{O})$ a topological realization of σ and any $j \in \mathbb{Z}/q\mathbb{Z}$ the restriction of f to $I_j := [x_j, x_{j+1}]$ lifts into Π as a homeomorphism $\widehat{f_j} : [x_j, x_{j+1}] \longrightarrow [\widehat{x}_{\sigma(j)}, \widehat{x}_{\sigma(j+1)}]$, where $\widehat{x}_{\sigma(j)} < \widehat{x}_{\sigma(j+1)}$, $\Pi(\widehat{x}_{\sigma(j)}) = x_{\sigma(j)}$ and $\Pi(\widehat{x}_{\sigma(j+1)}) = x_{\sigma(j+1)}$.
- It follows that $(f, \mathcal{O})$ is minimal iff $\widehat{x}_{\sigma(j)} < \widehat{x}_{\sigma(j+1)} < \widehat{x}_{\sigma(j)} + 1$ for each j.
- Or in other words $(f, \mathcal{O})$ is minimal only if for each j

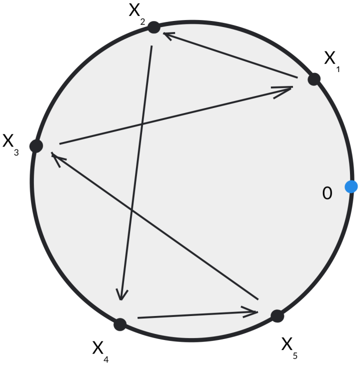
$$f(I_j) = [f(x_j), f(x_{j+1})] = [x_{\sigma(j)}, x_{\sigma(j+1)}] = \bigcup_{i \in [\sigma(j), \sigma(j+1))} I_i.$$

- Thus $(f, \mathcal{O})$ is minimal if and only if $\deg(f) = \deg(\sigma)$
- McMullen observed that a minimal realization of σ always exists:
- Take any q points with $0 < x_1 < \ldots < x_q < 1$ as $\mathcal{O}$ and let f be any map which for each j maps $[x_j, x_{j+1}]$ homeomorphically onto $[x_{\sigma(j)}, x_{\sigma(j+1)}]$.

Minimal realization of $\sigma = (1\ 2\ 4\ 5\ 3)$



(1 2 4 5 3)



Analysis of the botanics I

- We see immediately why $\sigma = (1\ 2\ 4\ 5\ 3)$ is not realized by $\mathbf{m}_2$. It has degree 3 and thus any realizing map must have topological degree at least 3.
- The four non-rotational period 4 combinatorics $\sigma_1 = (1\ 2\ 4\ 3)$, $\sigma_2 = (1\ 4\ 2\ 3)$, $\sigma_3 = (1\ 3\ 4\ 2)$ and $\sigma_4 = (1\ 3\ 2\ 4)$ are mutually conjugate by powers of the rotation $\rho = (1\ 2\ 3\ 4)$ and have degree 2. But only σ_1 is realized by $\mathbf{m}_2$. Why is this?
- Notice that 0 is a fixed point for any $\mathbf{m}_k$. Thus in general I_q must be mapped over itself, and in fact onto a larger interval in order for $\sigma \in \mathcal{C}_q$ to be realized by $\mathbf{m}_k$.
- This means for a $\sigma \in \mathcal{C}_q$ of degree d to be realized by $\mathbf{m}_d$ we must have $I_q = [x_q, x_1] \subset [x_{\sigma(q)}, x_{\sigma(1)}]$ or equivalently

$$\sigma(1) < \sigma(q).$$

Analysis of the botanics II

- We have thus arrived at

Proposition

A necessary condition for a combinatoric $\sigma \in \mathcal{C}_q$ to be realized by $\mathbf{m}_k$ is that

$$\deg(\sigma) \leq k \qquad and \qquad \sigma(1) < \sigma(q).$$

- The following theorem shows that these conditions are also sufficient.

Realization under m_d. I

Theorem (Zakeri and P.)

Let $\sigma \in \mathcal{C}_q$ be a q-cycle with $\deg(\sigma) = d \geq 2$.

- *If $\sigma(1) < \sigma(q)$ then σ has a realization under m_d and*
- *if $\sigma(1) > \sigma(q)$ then σ has a realization under m_{d+1}.*

In both cases the realisation is unique.

Realization under m_d. II

Theorem (Zakeri and P.)

Let $\sigma \in \mathcal{C}_q$ be a q-cycle with $\deg(\sigma) = d \geq 2$ *and let $k \geq d$. Then the number of realizations of σ under m_k is given by the binomial coefficient :*

$$\binom{q+k-d}{q} \qquad \text{if} \quad \sigma(1) < \sigma(q)$$

$$\binom{q+k-d-1}{q} \qquad \text{if} \quad \sigma(1) > \sigma(q).$$

- Note that for $d = 1$ (rotation cycles) and $k \geq 2$ this agrees with Goldbergs formula.
- I shall focus on the proof that a q-cycle $\sigma \in \mathcal{C}_q$ with $\deg(\sigma) = d \geq 2$ and $\sigma(1) < \sigma(q)$ is realised under m_d.

The transition matrix of σ.

Definition

The *transition matrix* of $\sigma \in \mathcal{C}_q$ is the $q \times q$ matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } j \in [\sigma(i), \sigma(i+1)) \\ 0 & \text{otherwise.} \end{cases}$$

- We may also view the transition matrix A geometrically:
- Let $(f, \mathcal{O})$ be a(ny) minimal realization of σ, where $\mathcal{O} = \{x_1, \ldots, x_q\}$ and as usual $0 < x_1 < \ldots < x_q < 1$. Then we saw above that

$$f(I_i) = \bigcup_{j \in [\sigma(i), \sigma(i+1))} I_j \qquad \text{for all } i,$$

where $I_j = [x_j, x_{j+1}]$.

The transition matrix of σ cont..

- It follows that the entries of the transition matrix $A = [a_{ij}]$ satisfy

$$a_{ij} = \begin{cases} 1 & \text{if } f(I_i) \supset I_j \\ 0 & \text{otherwise.} \end{cases}$$

- Since f is a covering map of degree d, *every column of the transition matrix A contains exactly d entries of* 1.
- The column stochastic matrix $\frac{1}{d} \cdot A$ describes a Markov chain with states $I_1, \ldots, I_q$, with the probability of going from I_j to I_i equal to $1/d$ if $I_j \subset f(I_i)$ and equal to 0 otherwise.

The Transition matrix and iteration

- Let A be the transition matrix of a q-cycle σ.
- Let $(f, \mathcal{O})$ be a minimal realization of σ with the partition intervals $I_1, \ldots, I_q$ as above.
- A straightforward induction shows that the ij-entry $a_{ij}^{(n)}$ of the power A^n is the number of times the n-th iterated image $f^{\circ n}(I_i)$ covers I_j or, equivalently, the number of connected components of $f^{-n}(I_j)$ in I_i.

Lemma

Let A be the transition matrix of $\sigma \in \mathcal{C}_q$ with $\deg(\sigma) \geq 2$. Then the power A^q has positive entries.

- This is shows that the transition matrix is irreducible.

A Perron – Frobenius Theorem

Theorem (Perron – Frobenius)

Let S be a $q \times q$ column stochastic matrix with the property that some power of S has positive entries. Then

(i) *S has a simple eigenvalue at $\lambda = 1$ and the remaining eigenvalues are in the open unit disk $\{\lambda : |\lambda| < 1\}$.*

(ii) *The eigenspace corresponding to $\lambda = 1$ is generated by a unique probability vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$ with $\ell_i > 0$ for all i.*

(iii) *The powers S^n converges to the matrix with identical columns $\boldsymbol{\ell}$ as $n \to \infty$.*

We immediately have :

Theorem

Let A be the transition matrix of $\sigma \in \mathcal{C}_q$ with $\deg(\sigma) = d \geq 2$. Then, there is a unique probability vector $\boldsymbol{\ell} \in \mathbb{R}^q$ such that $A\boldsymbol{\ell} = d\boldsymbol{\ell}$. Moreover, $\boldsymbol{\ell}$ has positive components and satisfies

$$\boldsymbol{\ell} = \lim_{n\to\infty} \frac{1}{d^n} A^n \mathbf{v}$$

for every probability vector $\mathbf{v} \in \mathbb{R}^q$.

We are now ready to prove the theorem :

Theorem

Let $\sigma \in \mathcal{C}_q$ be any q-cycle with $\deg(\sigma) = d \geq 2$ and with $\sigma(1) < \sigma(q)$. Then σ has a unique realization under $\mathbf{m}_d$.

PROOF:

- We are looking for a q-periodic orbit $\mathcal{O} = \{x_1, \ldots, x_q\}$ for $\mathbf{m}_d$, $0 < x_1 < \ldots < x_q < 1$ such that $\mathbf{m}_d(x_i) = x_{\sigma(i)}$ for all i.
- Assume for a moment that such $\mathcal{O}$ exists, let $I_i = [x_i, x_{i+1}]$, consider the lengths $\ell_i = |I_i|$, and form the probability vector

$$\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q) \in \mathbb{R}^q_+.$$

- Since $\mathbf{m}_d$ maps I_i homeomorphically onto $\bigcup_{j\in[\sigma(i),\sigma(i+1))} I_j$, we have

$$\sum_{j\in[\sigma(i),\sigma(i+1))} \ell_j = d\ell_i \qquad \text{for all } i. \tag{1}$$

- The q relations (1) can be written as

$$A\boldsymbol{\ell} = d\boldsymbol{\ell}, \tag{2}$$

  where A is the transition matrix of σ.
- By the Perron-Frobenius Theorem, this equation has a unique solution $\boldsymbol{\ell}$ which determines the lengths of the partition intervals $\{I_i\}$, hence the orbit $\mathcal{O}$ once we find x_1.
- To construct the orbit $\mathcal{O} = \{x_1, \ldots, x_q\}$, take the unique solution $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$ of (2) and define

$$\begin{cases} x_1 = \dfrac{1}{d-1} \displaystyle\sum_{j \in [1,\sigma(1))} \ell_j \\ x_i = x_1 + \displaystyle\sum_{j \in [1,i)} \ell_j \qquad \text{for } 2 \leq i \leq q. \end{cases} \tag{3}$$

- A few tedious computations shows that (3) works.

The higher degree cases

- Let $\sigma \in \mathcal{C}$. In order to describe the higher degree case $k > d = \deg(\sigma)$, we need some further notation.
- As before let A denote the transition matrix for σ.
- It can be shown that the diagonal of the $0-1$ matrix A contains precisely $d-1$ entries of 1.
- A diagonal entry say a_{ii} with value 1 corresponds to a fixed point for realizations.
- That is for any minimal realization $(f, \mathcal{O})$ of σ, the interval I_i contains a fixed point for f iff $a_{ii} = 1$, that is iff $f(I_i) \supset I_i$.
- In particular the q-th diagonal entry $a_{qq} = 1$ if and only if $\sigma(1) < \sigma(q)$.

The signature of σ.

We define

Definition

Let $A = [a_{ij}]$ be the transition matrix of $\sigma \in \mathcal{C}_q$ with $\deg(\sigma) = d$. The *signature* of σ is the integer vector formed by the main diagonal entries of A:

$$\mathbf{sig}(\sigma) = (a_{11}, \ldots, a_{qq}).$$

If $(f, \mathcal{O})$ is any realization of σ (minimal or not), and if $I_1, \ldots, I_q$ are the corresponding partition intervals, then I_i is called a *marked interval* if $a_{ii} = 1$.

- Let $\boldsymbol{p} = (p_1, \ldots, p_q) \in \mathbb{N}^q$ be a q-vector with non negative integer valued coordinates. And let $\mathbf{1}$ denote the q-vector of ones $\mathbf{1} = (1, \ldots, 1)$
- Let $P = \boldsymbol{p}^T \cdot \mathbf{1}$ be the $q \times q$ matrix with identical columns equal to $\boldsymbol{p}^T$ ...

The transformation matrix for non minimal realiztions.

- Then

$$B = A + P$$

can be regarded as the transition matrix for realizations $(f, \mathcal{O})$ of σ with winding p_i on interval I_i.
- That is lifts of f on $[x_i, x_{(i+1)}]$ to Π have homeomorphic images of the form $[\widehat{x}_{\sigma(i)}, \widehat{x}_{\sigma(i+1)}]$ where $\widehat{x}_{\sigma(i)} + p_i < \widehat{x}_{\sigma(i+1)} < \widehat{x}_{\sigma(i)} + p_i + 1$, $\Pi(\widehat{x}_{\sigma(i)}) = x_{\sigma(i)}$ and $\Pi(\widehat{x}_{\sigma(i+1)}) = x_{\sigma(i+1)}$.
- Then b_{ij} is the number of connected components of $f^{-1}(I_j)$ contained in I_i.
- The total sum of the elements in each column is

$$k = \deg(\sigma) + \sum_{j=1}^{q} p_j.$$

- Thus $\frac{1}{k}B$ is a column stochastic matrix.
- Applying the Perron-Frobenius theorem again we find that B has a unique simpel leading eigenvalue 1 and a unique corresponding positive probability eigen-vector.
- With this in place the following theorem is easily proved.

Theorem

If the diagonal element $b_{qq} = a_{qq} + p_q > 0$, then there are b_{qq} orbits for $\mathbf{m}_k$ realizing σ.

Workshop on Holomorphic Dynamics
- Iterated Monodromy groups and
Henon maps with a semi-neutral fixed point -
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http://thiele.ruc.dk/∼lunde/Monodromy/index.html