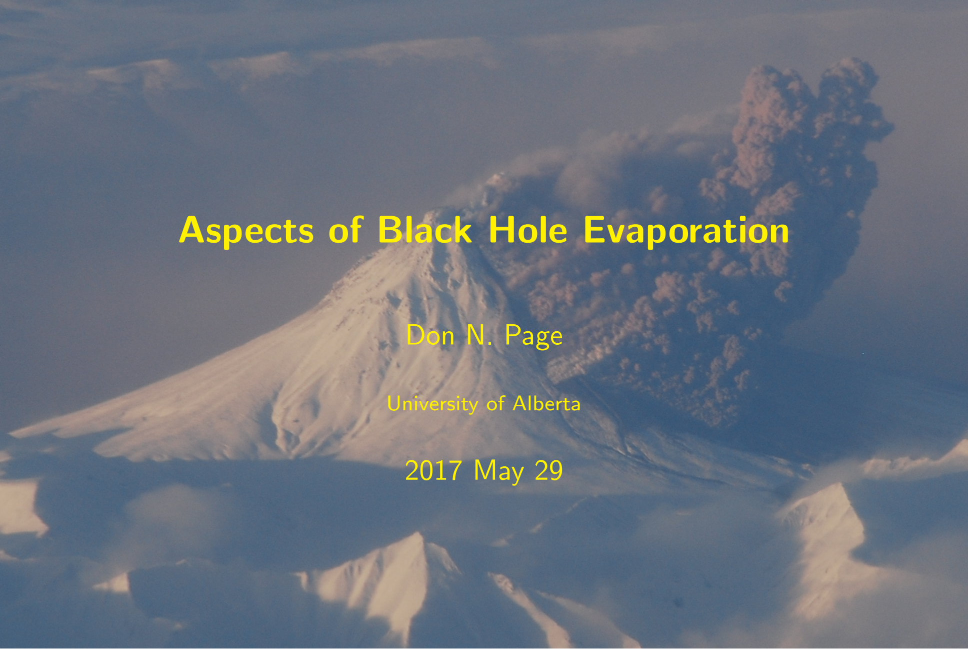

Aspects of Black Hole Evaporation

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Introduction

Classically black holes may be the simplest objects in the universe, but quantum mechanically they may be the most complex.

There are a number of timescales associated with a black hole. For a supermassive black hole of forty billion solar masses in our universe, these range from the quantum oscillation period of 9.3×10^{-92} seconds to the decay time of 2.3×10^{106} seconds, a ratio of 2.5×10^{197}. For a black hole of this mass in stable equilibrium in the microcanonical ensemble in anti-de Sitter spacetime of length scale not more than about 1.3×10^{31} light years, there is also a classical recurrence time of about $\exp(1.7 \times 10^{98})$ (roughly the 137th root of a googolplex) Planck times (or seconds or years), and a quantum recurrence time of roughly the exponential of this enormous number, $\sim \exp[10^{10^{100}\alpha}]$.

Planck units

To avoid the historical accidents of most conventional human units, here I shall express physical parameters as dimensionless multiples of Planck units defined in terms of the speed of light c, Planck's reduced constant $\hbar$, Newton's gravitational constant G, Coulomb's electric force constant $(4\pi\epsilon_0)^{-1}$, and Boltzmann's constant k_B as
Planck mass $\sqrt{\hbar c/G} = 2.176\,470(51) \times 10^{-8}$ kg,
Planck length $\sqrt{\hbar G/c^3} = 1.616\,229(38) \times 10^{-35}$ m,
Planck time $\sqrt{\hbar G/c^5} = 5.391\,16(13) \times 10^{-44}$ s,
Planck charge $\sqrt{4\pi\epsilon_0 \hbar c} = 1.875\,545\,956(41) \times 10^{-18}$ C,
Planck temperature $\sqrt{\hbar c^5/G}/k_B = 1.416\,808(33) \times 10^{32}$ K,
Planck energy $\sqrt{\hbar c^5/G} = 1.956\,114(45) \times 10^{9}$ J
$= 1.220\,910(29) \times 10^{19}$ GeV,
Planck power $c^5/G = 3.628\,37(17) \times 10^{52}$ W,
Planck energy density $c^7/(\hbar G^2) = 4.633\,25(44) \times 10^{113}$ J/m^3
$= 2.891\,85(27) \times 10^{132}$ eV/m^3 $= 0.849\,625(80) \times 10^{200}$ eV/Mpc^3.

Observed Physical Parameters in Planck Units

Elementary charge $e = \sqrt{\alpha} = [137.035\,999\,139(31)]^{-1/2} = [11.7062376167(13)]^{-1} = 0.085\,424\,543\,1148(98)$.

Proton mass $m_p = 1.672\,621\,898(21) \times 10^{-27}$ kg
$= 0.938\,272\,0813(58)$ GeV/c^2
$= 7.68502(18) \times 10^{-20} = 1.002\,420(23) \times 2^{-127/2}$ Planck masses.

Electron mass $m_e = 9.109\,383\,56(11) \times 10^{-31}$ kg
$= 0.000\,510\,998\,9461(31)$ GeV/c^2 $= 4.185\,394(96) \times 10^{-23}$.

Solar mass $M_\odot = 1.988\,49(9) \times 10^{30}$ kg $= 1476.625\,12$ m $c^2/G = 9.136\,24(21) \times 10^{37} = 0.536\,980(12) \times 2^{127} = 0.539\,582(12) m_p^{-2}$.

Supermassive black hole $M_2 \sim 4 \times 10^{10} M_\odot = 197\,020$ seconds $=$ 2.28032 days $= 3.65449 \times 10^{48}$ Planck masses.

One year $=$ 365.25 days $= 31\,557\,600$ s $= 5.853\,58(13) \times 10^{50}$.

Age of universe $t_0 = 13.80(4)$ Gyr $= 5.040(14)$ trillion days
$= 4.355(13) \times 10^{17}$ s $= 8.08(2) \times 10^{60}$ Planck times.

Cosmological constant $\Lambda = [0.998(32)]$ $(10\text{ Gyr})^{-2}$
$= [10.01(16)\text{ Gyr}]^{-2} = 1.002(32) \times 10^{-35}\text{ s}^{-2} = 2.91(9) \times 10^{-122}$
$= [0.998(32)](3\pi\, 5^{-3} 2^{-400}) \approx$ ten square attohertz.

Gibbons-Hawking entropy $S_\Lambda = 3\pi/\Lambda \approx 5^3 2^{400} \approx 3.23 \times 10^{122}$.

Black Hole Thermodynamics

After suggestions from Jacob Bekenstein that black holes have an entropy proportional to their horizon areas (which Stephen Hawking noted would be inconsistent with the supposed 'fact' that black holes can only absorb radiation), and after independent predictions by Yakov Zel'dovich and Alexei Starobinsky and others of us of quantum emission by rotating holes, Hawking found that even nonrotating black holes emit radiation and have a temperature (here given for a nonrotating uncharged black hole of mass M in asymptotically flat spacetime for simplicity, a Schwarzschild black hole, with event horizon radius $r_h = 2GM/c^2$),

$$T = \frac{\hbar}{2\pi c k_B}(\text{surface gravity}) = \frac{\hbar}{2\pi c k_B}\frac{GM}{r_h^2} = \frac{\hbar c^3}{8\pi c k_B GM} \equiv \frac{1}{8\pi M}.$$

This thermal Hawking radiation made Bekenstein's idea of entropy consistent and fixed the unknown constant of proportionality so that the dimensionless entropy (S/k_B, or S with $k_B = 1$) is

$$S = \frac{c^3}{4\hbar G}(\text{event horizon area } A) \equiv \frac{A}{4} = \frac{4\pi G}{\hbar c}M^2 \equiv 4\pi M^2.$$

Black Hole Evaporation Rates

D. N. Page, "Particle Emission Rates from a Black Hole: Massless Particles from an Uncharged, Nonrotating Hole," Phys. Rev. D **13**, 198 (1976).
D. N. Page, "Particle Emission Rates from a Black Hole. 2. Massless Particles from a Rotating Hole," Phys. Rev. D **14**, 3260 (1976).
D. N. Page, "Particle Emission Rates from a Black Hole. 3. Charged Leptons from a Nonrotating Hole," Phys. Rev. D **16**, 2402 (1977).

Photon and graviton emission from a Schwarzschild black hole:

- $dM/dt = -\alpha/M^2 \approx -0.000\,037\,474/M^2$.
- $d\tilde{S}_{\rm BH}/dt = -8\pi\alpha/M \approx -0.000\,941\,82/M$.
- $d\tilde{S}_{\rm rad}/dt \approx 0.001\,398\,4/M = -\beta d\tilde{S}_{\rm BH}/dt$.
- $\beta \equiv (d\tilde{S}_{\rm rad}/dt)/(-d\tilde{S}_{\rm BH}/dt) \approx 1.4847$.

Semiclassical Evolution

Black hole mass time dependence:

$$M(t) = (M_0^3 - 3\alpha t)^{1/3} = M_0(1 - t/t_{\text{decay}})^{1/3}.$$

Decay time for a large Schwarzschild black hole:

$$t_{\text{decay}} = \gamma M_0^3 \equiv \frac{1}{3\alpha} M_0^3 \approx 8895 M_0^3 \approx 1.159 \times 10^{67} \left(\frac{M_0}{M_\odot}\right)^3 \text{yr}.$$

Semiclassical Bekenstein-Hawking black hole entropy:

$$\tilde{S}_{\text{BH}}(t) = 4\pi M_0^2 \left(1 - \frac{t}{\gamma M_0^3}\right)^{2/3} \approx 4\pi M_0^2 \left(1 - \frac{t}{8895 M_0^3}\right)^{2/3}.$$

Semiclassical Hawking radiation entropy:

$$\begin{aligned} \tilde{S}_{\text{rad}}(t) &= 4\pi\beta M_0^2 \left[1 - \left(1 - \frac{t}{\gamma M_0^3}\right)^{2/3}\right] \\ &\approx 4\pi(1.4847) M_0^2 \left[1 - \left(1 - \frac{t}{8895 M_0^3}\right)^{2/3}\right]. \end{aligned}$$

Black Hole Information

Assuming local quantum field theory on a fixed dynamical spacetime background of an evaporating black hole, with its absolute event horizon that is the boundary of what can send signals to the outside, leads to loss of information from the exterior of the black hole. Hawking therefore predicted that when the black hole evaporates away, information would be lost from the universe, and a pure initial state would become a mixed final state:

S. W. Hawking, "Breakdown of Predictability in Gravitational Collapse," Phys. Rev. D **14**, 2460 (1976).

However, this argument depended on a semiclassical analysis, and the first paper to dispute it was the following:

D. N. Page, "Is Black Hole Evaporation Predictable?," Phys. Rev. Lett. **44**, 301 (1980).

After the issue lay in the doldrums for years, now most (but not all) opinion has switched to mine, and even Hawking has conceded.

Black Hole Firewalls

Interest in black hole information has surged recently with A. Almheiri, D. Marolf, J. Polchinski and J. Sully, "Black Holes: Complementarity or Firewalls?," JHEP 1302 (2013) 062.
They give a provocative argument that suggests that an "infalling observer burns up at the horizon" of a sufficiently old black hole, so that the horizon becomes what they called a "firewall."

Unitary evolution suggests that at late times the Hawking radiation is maximally entangled with the remaining black hole and neighborhood (including the modes just outside the horizon).
This further suggests that what is just outside cannot be significantly entangled with what is just inside.
But without this latter entanglement, an observer falling into the black hole should be burned up by high-energy radiation.

Time Dependence of Hawking Radiation Entropy

One cannot externally observe entanglement across the horizon. However, it should eventually be transferred to the radiation. Therefore, we would like to know the retarded time dependence of the von Neumann entropy of the Hawking radiation.

A. Strominger, "Five Problems in Quantum Gravity," Nucl. Phys. Proc. Suppl. **192-193**, 119 (2009) [arXiv:0906.1313 [hep-th]], has emphasized this question and outlined five candidate answers:

- bad question
- information destruction
- long-lived remnant
- non-local remnant
- maximal information return

I shall assume without proof maximal information return.

Assumptions

- Unitary evolution (no loss of information)
- Initial approximately pure state
  (e.g., $S_{\mathrm{vN}}(0) \sim S(\mathrm{star}) \sim 10^{57} \ll \tilde{S}_{\mathrm{BH}}(0) \sim 10^{77}$)
- Nearly maximal entanglement between hole and radiation
- Complete evaporation into just final Hawking radiation
- Nonrotating uncharged (Schwarzschild) black hole
- Initial black hole mass large, $M_0 > M_\odot$
- Massless photons and gravitons; other particles $m > 10^{-10}$ eV
- Therefore, essentially just photons and gravitons emitted

Arguments for Nearly Maximal Entanglement

D. N. Page, "Average Entropy of a Subsystem," Phys. Rev. Lett. **71**, 1291 (1993) [gr-qc/9305007].
"There is less than one-half unit of information, on average, in the smaller subsystem of a total system in a random pure state."

D. N. Page, "Information in Black Hole Radiation," Phys. Rev. Lett. **71**, 3743 (1993) [hep-th/9306083].
"If all the information going into gravitational collapse escapes gradually from the apparent black hole, it would likely come at initially such a slow rate or be so spread out . . . that it could never be found or excluded by a perturbative analysis."

Y. Sekino and L. Susskind, "Fast Scramblers," JHEP **0810**, 065 (2008) [arXiv:0808.2096 [hep-th]], conjecture:

- The most rapid scramblers take a time logarithmic in the number of degrees of freedom.
- Black holes are the fastest scramblers in nature.

These conjectures support my results using an average over all pure states of the total system of black hole plus radiation.

von Neumann Entropies of the Radiation and Black Hole

Take the semiclassical entropies $\tilde{S}_{\mathrm{rad}}(t)$ and $\tilde{S}_{\mathrm{BH}}(t)$ to be approximate upper bounds on the von Neumann entropies of the corresponding subsystems with the same macroscopic parameters.

Therefore, the von Neumann entropy of the Hawking radiation, $S_{\mathrm{vN}}(t)$, which assuming a pure initial state and unitarity is the same as the von Neumann entropy of the black hole, should not be greater than either $\tilde{S}_{\mathrm{rad}}(t)$ or $\tilde{S}_{\mathrm{BH}}(t)$.

Take my 1993 results as suggestions for the *Conjectured Anorexic Triangle Hypothesis (CATH)*:

Entropy triangular inequalities are usually nearly saturated.

This leads to the assumption of nearly maximal entanglement between hole and radiation, so $S_{\mathrm{vN}}(t)$ should be near the minimum of $\tilde{S}_{\mathrm{rad}}(t)$ and $\tilde{S}_{\mathrm{BH}}(t)$.

Time of Maximum von Neumann Entropy

Since the semiclassical entropy $\tilde{S}_{\rm rad}(t)$ is monotonically increasing with time, and since the semiclassical entropy $\tilde{S}_{\rm BH}(t)$ is monotonically decreasing with time, the maximum von Neumann entropy is at the crossover point, at time

$$t_* = \epsilon t_{\rm decay} \approx 0.5381\, t_{\rm decay} \approx 4786\, M_0^3 \approx 6.236 \times 10^{66} (M_0/M_\odot)^3 {\rm yr},$$

with

$$\epsilon \equiv 1 - [\beta/(\beta+1)]^{3/2} \approx 0.5381,$$

at which time the mass of the black hole is

$$M_* = [\beta/(\beta+1)]^{1/2} M_0 \approx 0.7730\, M_0,$$

and its semiclassical Bekenstein-Hawking entropy $4\pi M^2$ is

$$\tilde{S}_{\rm BH*} = [\beta/(\beta+1)] \tilde{S}_{\rm BH}(0) \approx 0.5975\, \tilde{S}_{\rm BH}(0).$$

Maximum von Neumann Entropy of the Hawking Radiation

At the time t_* when $\tilde{S}_{\rm rad}(t) = \tilde{S}_{\rm BH}(t)$, the von Neumann entropy of the radiation and of the black hole is maximized and has the value

$$S_* \equiv S_{\rm vN}(t_*) = \tilde{S}_{\rm rad}(t_*) = \tilde{S}_{\rm BH}(t_*) = \left(\frac{\beta}{\beta+1}\right) 4\pi M_0^2 \approx 0.5975\, \tilde{S}_{\rm BH}(0)$$
$$= 0.5975(4\pi M_0^2) \approx 7.509 M_0^2 \approx 6.268 \times 10^{76} (M_0/M_\odot)^2.$$

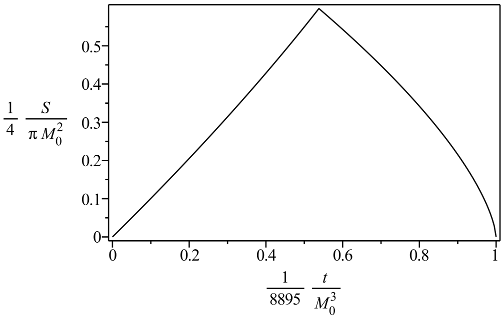
Note that this maximum of the von Neumann entropy is about 19.5% greater than half the original semiclassical Bekenstein-Hawking entropy of the black hole. The time t_* for the maximum von Neumann entropy is about 0.8324 times the time $t_{1/2} = (1 - 2^{-3/2}) t_{\rm decay} \approx 0.6464\, t_{\rm decay} \approx 1.201\, t_*$ for the black hole to lose half its area and semiclassical Bekenstein-Hawking entropy.

Time Dependence of the Entropy of the Hawking Radiation

The von Neumann entropy of the Hawking radiation $S_{\mathrm{vN}}(t)$ from a large nonrotating uncharged black hole is very nearly the semiclassical radiation entropy $\tilde{S}_{\mathrm{rad}}(t)$ for $t < t_*$ and is very nearly the Bekenstein-Hawking semiclassical black hole entropy $\tilde{S}_{\mathrm{BH}}(t)$ for $t > t_*$, or, using the Heaviside step function $\theta(x)$,

$$\begin{aligned}
S_{\mathrm{vN}}(t) &\approx 4\pi\beta M_0^2\left[1-\left(1-\frac{t}{t_{\mathrm{decay}}}\right)^{2/3}\right]\theta(t_*-t) \\
&+ 4\pi M_0^2\left(1-\frac{t}{t_{\mathrm{decay}}}\right)^{2/3}\theta(t-t_*) \\
&\approx 4\pi(1.4847)M_0^2\left[1-\left(1-\frac{t}{8895M_0^3}\right)^{2/3}\right]\theta(4786M_0^3-t) \\
&+ 4\pi M_0^2\left(1-\frac{t}{8895M_0^3}\right)^{2/3}\theta(t-4786M_0^3).
\end{aligned}$$

Plot of Hawking Radiation Entropy vs. Time



Corrections to the Approximate Entropy Formula

- Fluctuations in $M(t)$: $\Delta S = O(M_0)$.
- Fluctuations in $\mathbf{x}(t)$: $\Delta S = O(M_0)$.
- Entropy of black hole motion: $\Delta S = O(\ln M_0)$.
- Nonmaximal entanglement: $\Delta S = O(1)$.
- Entropy near black hole: $\Delta S = O(1)$.
- Fuzziness of t boundary: $\Delta S = O(1)$.

Summary of the Time Dependence of the von Neumann Entropy

Under the assumptions that a Schwarzschild black hole of initial mass $M_0 > M_\odot$ (too massive to emit anything but photons and gravitons) starts in nearly a pure quantum state and decays away completely by a unitary process while being nearly maximally scrambled at all times, the von Neumann entropy of the Hawking radiation increases up to a maximum of $S_* \approx 0.5975(4\pi M_0^2) \approx 7.509 M_0^2 \approx 6.268 \times 10^{76}(M_0/M_\odot)^2$ at time $t_* \approx 0.53810\, t_{\mathrm{decay}} \approx 4786 M_0^3 \approx 6.236 \times 10^{66}(M_0/M_\odot)^3 \mathrm{yr}$ and then decreases back down to near zero.

If the black hole starts in a maximally mixed state ($f = 1$, so $S_0 \equiv f \tilde{S}_{\mathrm{BH}}(0) = \tilde{S}_{\mathrm{BH}}(0) = 4\pi M_0^2$), the von Neumann entropy of the Hawking radiation increases from zero up to a maximum of $S_{*'} \approx 1.1951(4\pi M_0^2) \approx 15.018 M_0^2 \approx 1.254 \times 10^{77}(M_0/M_\odot)^2$ at $t_{*'} \approx 0.91384\, t_{\mathrm{decay}} \approx 8129 M_0^3 \approx 1.059 \times 10^{67}(M_0/M_\odot)^3 \mathrm{yr}$ and then decreases back down to $S_0 = 1.049 \times 10^{77}(M_0/M_\odot)^2$.

Black Holes in Anti-de Sitter Spacetime

So far I have been talking about black holes that evaporate away in asymptotically flat spacetime. However, in anti-de Sitter spacetime with its negative cosmological constant, if one imposes either thermal or reflecting boundary conditions at the timelike conformal boundary at radial infinity, sufficiently large black holes can be stable in either the canonical or microcanonical ensemble:

S. W. Hawking and D. N. Page, "Thermodynamics of Black Holes in Anti-de Sitter Space," Commun. Math. Phys. **87**, 577 (1983).

$$ds^2 = -\left(1 + \frac{r^2}{b^2} - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{r^2}{b^2} - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

The canonical ensemble gives a stable black hole if $T \geq \sqrt{3}/(2\pi b)$ and $M \geq 2b/\sqrt{27} = 2.466 \times 10^{21}\, M_\odot (b/\mathrm{Gyr})$, $b = (-3/\Lambda)^{1/2}$. The microcanonical ensemble gives a stable black hole if the total energy of the black hole plus photon and graviton radiation is $E \geq 0.274352\, b^{3/5}$ in Planck units, or $0.002178\, M_\odot (b/\mathrm{Gyr})^{3/5}$.

Classical Recurrence Time in Anti-de Sitter

If r_h is the radius of the horizon of Schwarzschild-anti-de Sitter, the black hole mass M and entropy S are

$$M = \frac{r_h}{2} + \frac{r_h^3}{2b^2}, \; S = \pi r_h^2.$$

The number of quantum states of a black hole of roughly this size is approximately e^S. This multiplied by the time for light to travel a distance comparable to the size of the black hole, say r_h, is the classical recurrence time, the expected time to return to near the initial state if the black hole were undergoing a sequence of classical transitions. E.g., if the black hole were represented by a set of roughly S classical qubits (ignoring factors of $\ln 2$) that each could take the values 0 or 1, this would be the expected time until the black hole returned to the same sequence of qubits.

Leonard Susskind has argued that black holes formed by collapse from smooth initial conditions remain free of firewalls for a time at least of the order of the classical recurrence time, assuming they are kept from evaporating.

Quantum Recurrence Time in Anti-de Sitter

If one wants not just the microscopic classical configuration of a black hole of entropy S to return to near some previous value, but also the quantum state, one needs all of the amplitudes for all of the roughly e^S basis states to return to very near the same values, so that the sum of the absolute squares of the differences of these amplitudes are small. This much stronger constraint takes a quantum recurrence time, which for black holes is estimated to be of the order of e^{e^S}, doubly exponential in the entropy (or in the number of qubits representing the black hole).

Susskind has argued that the *complexity* of a black hole quantum state, which is defined as the minimal number of a certain set of quantum gates needed to produce the state from a simple fiducial state, grows roughly linearly with time for a time $\sim r_h e^S$ until it reaches near the maximum possible value, e^S, and then the complexity stays near the maximum for a time $\sim r_h e^{e^S}$ before fluctuating down to a low value again.

Times Associated with a Black Hole

Let me collect together a number of different times associated with a black hole of mass M and give Planck-unit numbers for $M_1 = M_\odot = 9.136 \times 10^{37}$, the solar mass (though stars of this mass are believed to be too small to form black holes), and for $M_2 = 4 \times 10^{10} M_\odot = 3.654 \times 10^{48}$, which is roughly the highest estimate for any supermassive black hole.

Roughly the shortest timescale I can think of is the quantum oscillation period, $t_Q = 2\pi/M$:

$$t_{Q1} = 6.8772 \times 10^{-38} = 3.7076 \times 10^{-81} \text{ seconds},$$
$$t_{Q2} = 1.7193 \times 10^{-48} = 9.2690 \times 10^{-92} \text{ seconds}.$$

Next shortest might be the time to travel one Schwarzschild radius $r_h = 2M$ (in asymptotically flat spacetime, $\Lambda = 0$), what might be crudely called a light-crossing time:

$$t_{R1} = 1.8272 \times 10^{38} = 9.8510 \times 10^{-6} \text{ seconds},$$
$$t_{R2} = 7.3090 \times 10^{48} = 394\,040 \text{ seconds} = 4.5606 \text{ days}.$$

More Times Associated with a Black Hole

For $M_1 = M_\odot = 9.136 \times 10^{37}$, $M_2 = 4 \times 10^{10} M_\odot = 3.654 \times 10^{48}$, scrambling times $t_S = 4M \ln S = 4M \ln\left(4\pi M^2\right)$ are

$t_{S1} = 6.4811 \times 10^{40} = 0.0034941$ seconds,
$t_{S2} = 3.3062 \times 10^{51} = 1.7824 \times 10^{8}$ seconds $= 5.6481$ years.

Oscillation periods $t_O = 2\pi b$ of geodesic motion in largest scale anti-de Sitter spacetimes in which energies $E = M_1$ and $E = M_2$ give stable microcanonical ensembles, $t_O = 54.2415 E^{5/3}$,

$t_{O1} = 1.005 \times 10^{65} = 5.420 \times 10^{21}$ seconds $= 1.717 \times 10^{14}$ years,
$t_{O2} = 4.703 \times 10^{82} = 2.535 \times 10^{39}$ seconds $= 8.034 \times 10^{31}$ years.

Times to reach the maximum von Neumann entropy of the Hawking radiation, the so-called Page time, $t_P = 4786 M^3$, are

$t_{P1} = 3.650 \times 10^{117} = 1.968 \times 10^{74}$ seconds $= 6.325 \times 10^{66}$ years,
$t_{P2} = 2.336 \times 10^{149} = 1.259 \times 10^{106}$ seconds $= 3.991 \times 10^{98}$ years.

Hawking evaporation decay times $t_D = 8895 M^3$ are

$t_{D1} = 6.783 \times 10^{117} = 3.658 \times 1074$ seconds $= 1.159 \times 10^{67}$ years,
$t_{D2} = 4.341 \times 10^{149} = 2.341 \times 10^{106}$ seconds $= 7.417 \times 10^{98}$ years.

Recurrence Times for a Black Hole in a Stable Microcanonical Ensemble in Anti-de Sitter Spacetime with Reflecting Boundary Conditions at Infinity to Prevent the Hawking Evaporation of the Black Hole

For $M_1 = M_\odot = 9.136 \times 10^{37}$, $M_2 = 4 \times 10^{10} M_\odot = 3.654 \times 10^{48}$, the classical recurrence times (in Planck units or in units of the Schwarzschild radius; it doesn't matter to the precision given for the exponents), $t_{CR} \sim e^S = e^{4\pi M^2}$, are

$t_{CR1} \sim e^{1.049 \times 10^{77}} \sim 10^{4.555 \times 10^{76}}$,
$t_{CR2} \sim e^{1.678 \times 10^{98}} \sim 10^{7.289 \times 10^{97}}$,
roughly a googolplex ($10^{10^{100}}$) raised to the power of the fine-structure constant ($\alpha \approx 0.0072973525664$).

The quantum recurrence times, $T_{QR} \sim e^{t_{CR}} \sim e^{e^S} \sim e^{e^{4\pi M^2}}$, are

$t_{QR1} \sim e^{e^{1.049 \times 10^{77}}} \sim 10^{10^{4.555 \times 10^{76}}}$,
$t_{QR2} \sim e^{e^{1.678 \times 10^{98}}} \sim 10^{10^{7.289 \times 10^{97}}}$, very roughly $(10^{10^{100}})^{\alpha}!$.

Conclusions

Classically a vacuum spherical black hole is very simple, described by the Schwarzschild metric when $\Lambda = 0$ or by the Schwarzschild-anti-de Sitter metric when $\Lambda = -3/b^2 < 0$.

However, a quantum black hole has a complicated behavior, with many different timescales: the quantum oscillation time $2\pi/M$, the light-crossing time $2M$, the scrambling time $4M \ln(4\pi M^2)$, the largest AdS period for a microcanonical ensemble that is about $54.24\, M^{5/3}$, the so-called Page time $4786\, M^3$, the Hawking decay time $8895\, M^3$, the classical recurrence time $e^{4\pi M^2}$, and the quantum recurrence time $e^{e^{4\pi M^2}}$. For a supermassive black hole of forty billion solar masses, these times range from less than 10^{-91} seconds to very roughly the factorial of the 137th root of a googolplex.