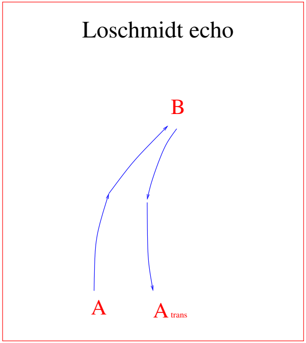
Quantum bound to chaos as a limit to wave scattering

IHES 2017

1. Semiclassical analysis

Quantum Lyapunov Exponent:

$$A_{trans} = \left\{ e^{i\frac{t}{\hbar}H} e^{i\delta B} e^{-i\frac{t}{\hbar}H} \right\}^{\dagger} A \left\{ e^{i\frac{t}{\hbar}H} e^{i\delta B} e^{-i\frac{t}{\hbar}H} \right\}$$



go $\rightarrow$ small kick $\rightarrow$ back

$$F_1 \propto \mathbf{Tr}[A^2] - \mathbf{Tr}[AA_{trans}] \propto e^{\lambda t}$$ with λ independent of A, B

Within linear response:

$$F_1 \propto \mathrm{Tr}\left[A^2 - AA_{trans}\right] \propto -\mathrm{Tr}\left\{[e^{i\frac{t}{\hbar}H}Be^{-i\frac{t}{\hbar}H}, A]^2\right\} = -\mathrm{Tr}\left\{[B(t), A]^2\right\}$$

We need the operator A to be localized. MSS chose

$$A \propto e^{-\frac{\beta}{4}H}\ W\ e^{-\frac{\beta}{4}H}$$

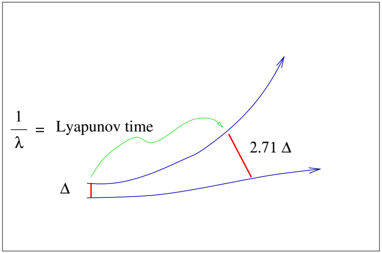
$$e^{\lambda t} \propto -\mathrm{Tr}\left\{e^{i\frac{t}{\hbar}H}Be^{-i\frac{t}{\hbar}H}e^{-\frac{\beta}{4}H}We^{-\frac{\beta}{4}H}e^{i\frac{t}{\hbar}H}Be^{-i\frac{t}{\hbar}H}e^{-\frac{\beta}{4}H}We^{-\frac{\beta}{4}H}\right\}_{connected}$$

A four-time function

Quantum bounds inspired by String Theory:

Bound on chaos Maldacena, Shenker and Stanford

$$\lambda \leq \frac{2\pi T}{\hbar}$$



A symptom:

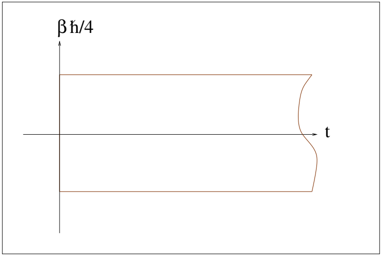
Solvable ‘Black Holes’ saturate the bound

A fermionic quantum spin glass model (SYK) does too

and has zero temperature entropy and linear specific heat capacity

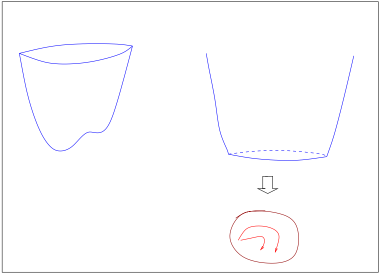
$$H = \sum J_{i_1,\ldots,i_q} \hat{\eta}_{i_1} \ldots \hat{\eta}_{i_q}$$

The proof is simple but not intuitive



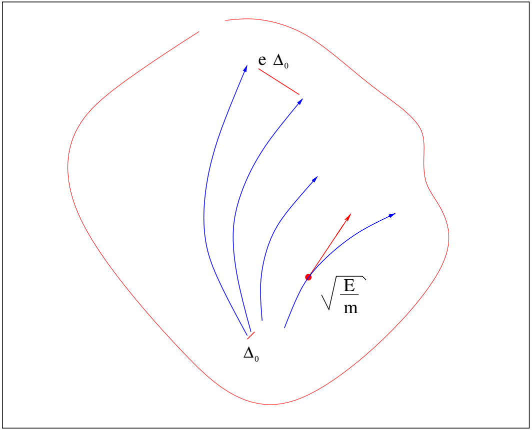
Semiclassical analysis

Semiclassical $\rightarrow$ needs low T to be interesting $\rightarrow$ needs structure at $T = 0$



$$H = \frac{1}{2m}\, p_i\, g^{ij}(q)\, p_j \qquad \Rightarrow \textit{free motion on a curved surface}$$

Motion is along geodesics :

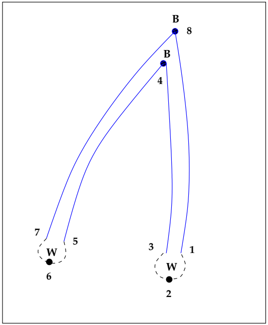


A direct semiclassical calculation: how does $\hbar \neq 0$ kick in?

$$e^{\lambda t} \propto -\mathrm{Tr}\left\{ e^{i\frac{t}{\hbar}H} B e^{-i\frac{t}{\hbar}H} e^{-\frac{\beta}{4}H} W e^{-\frac{\beta}{4}H} e^{i\frac{t}{\hbar}H} B e^{-i\frac{t}{\hbar}H} e^{-\frac{\beta}{4}H} W e^{-\frac{\beta}{4}H} \right\}_{connected}$$

insert

$$\boxed{\langle q' | e^{\pm i\frac{t}{\hbar}H} | q \rangle \sim A^{\pm}(q', q) e^{\pm \frac{i}{\hbar} S^c(q', q)}}$$



N degrees of freedom

- Geodesic deviation (*in space*) $\delta \ddot{q}_i = G(R, \dot{q}, \delta q_i)$

$$\Delta(\ell) \sim \Delta(0) e^{\ell/\ell_N}$$

where ℓ_N, the *Lyapunov length*. We expect that in a system with a good thermodynamic limit, ℓ_N scales with the size as

$$\ell_N = \ell_o \sqrt{N}$$

with ℓ_o a length that has a finite thermodynamic limit.

- **In time:**

$$\lambda = \frac{\text{speed}}{\ell_N} = \frac{\sqrt{2E/m}}{\ell_N} \quad \Rightarrow \quad \lambda = \frac{\sqrt{2\mathcal{E}/m}}{\ell_o}$$

what counts is the energy *per degree of freedom*

Let us compare Lyapunov and thermal deBroglie lengths $\ell_{dB} = \left(\frac{4\pi^2\hbar^2}{Tm}\right)^{1/2}$

$$\ell_N \div \sqrt{N}\ell_{dB}$$

n.b. the scaling with N

$$\ell_0 \div \ell_{dB} = \left(\frac{4\pi^2\hbar^2}{Tm}\right)^{1/2}$$

$$\frac{\sqrt{2E/mN}}{\lambda} \div \left(\frac{\hbar^2}{Tm}\right)^{1/2}$$

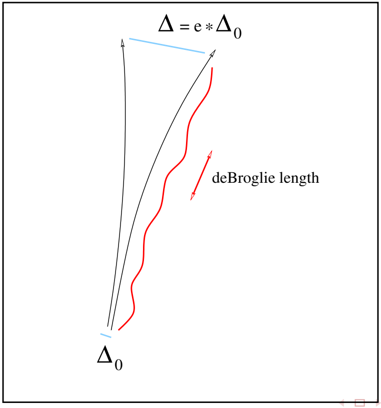
$$\frac{\sqrt{T/m}}{\lambda} \div \left(\frac{\hbar^2}{Tm}\right)^{1/2}$$

$$\frac{1}{\beta\hbar} \div \lambda$$

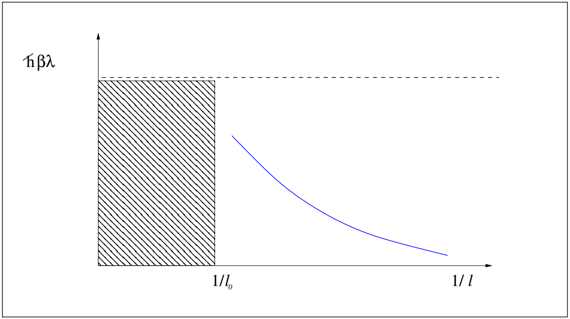
Both lengths are comparable if:

$$\ell_o \sim \ell_{dB} \quad \Rightarrow \quad \hbar\beta\lambda \sim 1$$

i.e. a wavefront does not scatter over lengths shorter than its wavelength



Quantum effects act when the semiclassical calculation breaks down



... and presumably localization effects appear

Note the roles of *a)* $T = 0$ entropy, *b)* criticality (many ℓ_o) *c)* localisation.

2. Glass (pre)history

The grandfather of SYK:

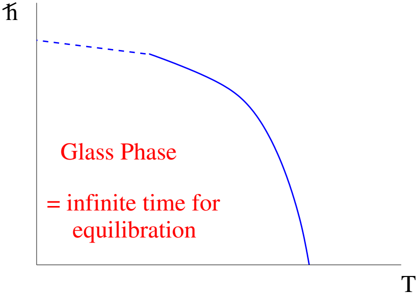
a model for structural glasses *if* $q > 2$

- **Classical** $H = \sum J_{i_1,\dots,i_q} \eta_{i_1} \dots \eta_{i_q}$

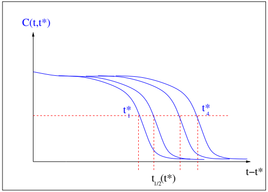
η_i are c-numbers constrained to $\eta_i = \pm 1$ or the sphere $\sum \eta_i^2 = N$, etc

- **Quantum** $H = \sum \hat{p}_i^2 + \sum J_{i_1,\dots,i_q} \hat{\eta}_{i_1} \dots \hat{\eta}_{i_q}$

$$[\hat{\eta}_i, \hat{p}_j] = i\hbar\, \delta_{ij}$$



The slow part has a pseudo reparametrization invariance (similar but not identical to the one in imaginary time), obtained by neglecting the time-derivative. **We know some things about its meaning and role ... with gaps.**



3. A model

Bar-Lev, Biroli, JK, Reichman

- Start with $H = \sum J_{i_1,\dots,i_q} \eta_{i_1} \dots \eta_{i_q}$ e.g. $\sum_i \eta_i^2 = N$

- Consider its stochastic dynamics at temperature T_s:

$$\dot{\eta}_i = -\frac{\partial H}{\partial \eta_i} + \sqrt{2 T_s}\, \zeta_i$$

- The probability distribution $P(\eta, t)$ evolves with:

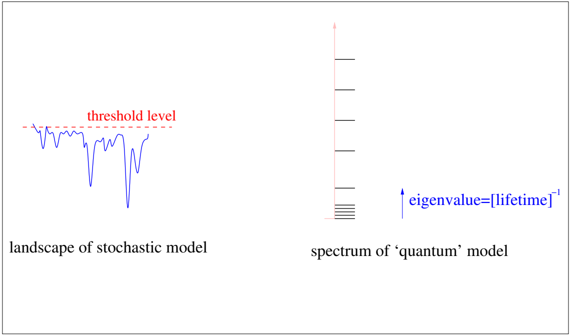
$$\dot{P}(\eta, t) = -\mathcal{L}\, P(\eta, t)$$

- Transform $\mathcal{L}$ into a hermitean operator:

$$\hat{H}_q = e^{\beta_s H/2}\, \mathcal{L}\, e^{-\beta_s H/2} = \frac{2}{T_s} \sum_i \left[-\frac{T_s^2}{2} \frac{\partial^2}{\partial \eta_i^2} + \frac{1}{8} \left(\frac{\partial H}{\partial \eta_i} \right)^2 - \frac{T_s}{4} \frac{\partial^2 H}{\partial \eta_i^2} \right]$$

- Now take seriously H_q as a quantum operator (i.e. SUSY Quantum Mechanics)

A one-to-one correspondence between thermodynamic phases and eigenvalues/eigenvectors



Eigenvalue= inverse lifetime $\Rightarrow$ entropy= log # metastable phases

Zero temperature entropy $\rightarrow$ log # number of states with diverging lifetimes

- **Entropy goes as non trivial power law (>1, however)**
- **Zero temperature critical point**
- **We know in full detail the structure, number and distribution of near-ground states**
- **'Real time' dynamics exactly solvable. Characteristic time $\sim (\textit{temperature})^2$ (for $q = 2$)**
- **We have not yet calculated λ**

- Simple-minded elementary results
- ‘Glassy’ ideas applied to the field theory systems
- Applications *to* glassy systems of some field theory techniques

A nice example:

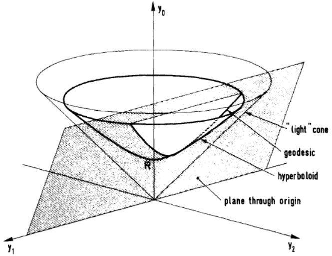


Fig. 1. y_0, y_1, y_2 are Minkowskian rectangular coordinates in the three-dimensional embedding space. The upper hyperboloid $y_0^2 - y_1^2 - y_2^2 = R^2$ is the pseudosphere. At infinity the pseudosphere approaches the light cone through the origin, forming the boundary at infinity. The intersection of the hyperboloid with a plane through the origin is a geodesic, and all geodesics are formed in this manner.