# The melonic universality class

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# Holography

Holography: in a theory with gravity one must have a correspondence between a volume of space and the boundary enclosing it (black hole entropy).

Gravity in $AdS_{d+1} \times \mathrm{Compact} \Leftrightarrow CFT_d$.

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Find interesting examples of (near) $CFT_1$!

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Exhibit non trivial scaling dimension (conformal weight $\Delta$) in the infrared

$$G(\tau_1 - \tau_2) \sim \frac{1}{|\tau_1 - \tau_2|^{2\Delta}} \ , \qquad \tilde{G}(\omega) = \int_{-\infty}^{\infty} d\tau \ \frac{1}{|\tau|^{2\Delta}} e^{\imath \omega \tau} \sim |\omega|^{2\Delta - 1}$$

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Feynman graph with

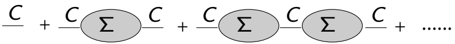
- $E$ edges $\to \Lambda^{(2\Delta-1)E}$
- $V$ vertices, $E - V + 1$ loop integrals $\to \Lambda^{E-V+1}$

$$\Lambda^{2E\Delta - V} = \Lambda^{\overbrace{2E}^{qV}\Delta - V} = \Lambda^{(\Delta q - 1)V} \Rightarrow^{marginal} \Delta q = 1 , \qquad G(\tau) \sim \frac{1}{|\tau|^{\frac{2}{q}}}$$

# The two point function

Schwinger Dyson equation:

$$G = \frac{1}{\underbrace{C^{-1}}_{\omega^{\alpha\geq 0}} - \Sigma}$$



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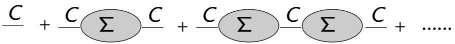


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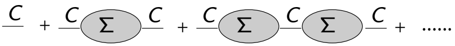


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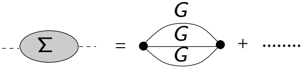
$\Sigma$ = Complicated function$(G)$

$$\Sigma(\tau_1 - \tau_2) = \underbrace{J \times 0}_{\text{massless, no tadpole}} + J^2 \left[ G(\tau_1 - \tau_2) \right]^{q-1} + \dots$$



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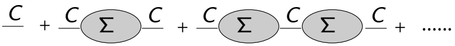


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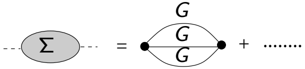


Simple equation if the rest is suppressed: $\delta(\tau) = J^2 \int du \, G(u-\tau)[G(u)]^{q-1}$

# The $\imath\epsilon$ prescription

$$G(\tau) = b\frac{\mathrm{sgn}(\tau)}{|\tau|^{\frac{2}{q}}} \qquad \delta(\tau) = J^2 b^q \int du \, \frac{\mathrm{sgn}(u-\tau)}{|u-\tau|^{\frac{2}{q}}} \left( \frac{\mathrm{sgn}(u)}{|u|^{\frac{2}{q}}} \right)^{q-1} \sim \frac{J^2 b^q}{|\tau|}$$

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**Theorem**

*Let $A^{\epsilon}(\tau)$ be:*

$$A^{\epsilon}(\tau) = J^2 \int du \, G^{\epsilon}(u-\tau)[G^{\epsilon}(u)]^{q-1} \ .$$

*Then, for any $\epsilon > 0$, $A^{\epsilon}(\tau)$ is a distribution (either on the space of test function with compact support or on Schwartz functions) and, in the sense of distributions,*

$$\lim_{\epsilon \to 0} A^{\epsilon}(\tau) = \delta(\tau) \ .$$

# Melonic theories

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Yes: "Melonic" theories built on random tensors!

# Random tensors

Random Tensors generalize Random Matrices to higher dimension.

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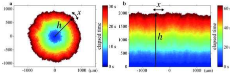
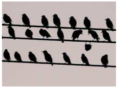
- Growing interfaces fluctuations




- Spacing between perched birds (parked cars)

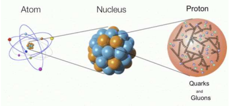


- Distinguish "signal" from "noise"


A. Chakraborty (2011) identify DNA sectors in HIV that rarely undergo multiple mutations

# Quantum Chromodynamics and random matrices

Strong interaction: quarks bind into protons and neutrons which form nuclei.




$\mathcal{SU}(3)$ gauge theory with 6 flavors of quarks:

$$L = \sum_f \bar{\psi}_f (\imath \gamma^\mu D_\mu - m) \psi_f^{\swarrow \text{3 components}} - \frac{1}{2} \mathrm{Tr}\left[ G_{\mu\nu} G^{\mu\nu} \right] ,$$

$$D_\mu = \partial_\mu - \imath z A_\mu^{\swarrow \substack{\text{3×3 hermitian} \\ \text{traceless matrix}}} , \quad G_{\mu\nu} = \frac{\imath}{z} \left[ D_\mu , D_\nu \right]$$

Endgame: expectations of gauge invariant observables (the Wilson loop ↔ quark confinement):

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [d\bar{\psi} d\psi dA] \; \mathcal{O} e^{\imath L}$$

# The $1/N$ expansion in random matrices

Random $N \times N$ matrices:

- Feynman expansion in embedded graphs (combinatorial maps) $\leftrightarrow$ discretized surfaces.
- Have built in scales: the size of the matrix, $N$ (number of degrees of freedom).

Canonical framework for the study of random surfaces (1980 onward): string theory, quantum gravity in $D = 2$, integrability, conformal field theory, invariants of algebraic curves, etc.

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$1/N$ expansion indexed by the genus

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## $1/N$ in 2010/2011.

150+ papers: A Baratin, D. Benedetti, J. Ben Geloun, V. Bonzom, S Carrozza, S. Dartois, T. Delepouve, A. Eichhorn, L. Freidel, T. Koslowski, T. Krajewski, V. Lahoche, L. Lioni, D. Oriti, V. Rivasseau, A. Riello, J.P. Ryan, D.O. Samary, M. Smerlak, L. Sindoni, A. Tanasa, F. Vignes-Tourneret, etc.

# 0–dimensional gauge theories

Random matrices ⇔ 0–dimensional gauge theories for at most two copies of the unitary (orthogonal, etc) group:

$$Z = \int [dA] \; e^{N\mathrm{Tr}[S(A)]} \,, \qquad A \to UAU^{\dagger} \quad U \in \mathcal{U}(N)$$

$$Z = \int [dMdM^{\dagger}] \; e^{N\mathrm{Tr}[S(MM^{\dagger})]} \,, \qquad M \to UMV^{\dagger} \,, \; M^{\dagger} \to VM^{\dagger}U^{\dagger} \quad U \in \mathcal{U}(N) \,, \; V \in \mathcal{U}(N)$$

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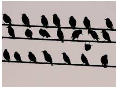
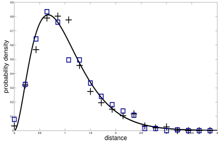
So where are the birds?

$$S(A) = -A^2/2 \xrightarrow{\text{eigenvalues}} Z = \int \Big(\prod_i d\lambda_i\Big) \underbrace{\Big(\prod_{i<j} |\lambda_i - \lambda_j|^2\Big)}_{\text{eigenvalue repulsion}} \; \underbrace{e^{-\frac{N}{2}\sum_i \lambda_i^2}}_{\text{confining potential}}$$






Gap distribution at large $N$ fits the distribution of the spacing between perched birds (squares) or parked cars (crosses)! (Šeba 2013)

## Tensor models

*Tensor Models are 0–dimensional gauge theories with gauge group the tensor product of $D \geq 3$ copies of the unitary (orthogonal, etc.) group.*

The field → rank $D$ complex tensor (no symmetry) transforming in the external tensor product of $D$ fundamental representations of $\mathcal{U}(N)^{\otimes D}$:

$$T'_{b^1\dots b^D} = \sum_a U^{(1)}_{b^1a^1} \dots U^{(D)}_{b^Da^D}\, T_{a^1\dots a^D} \;, \qquad \bar{T}'_{p^1\dots p^D} = \sum_q \bar{U}^{(1)}_{p^1q^1} \dots \bar{U}^{(D)}_{p^Dq^D}\, \bar{T}_{q^1\dots q^D}$$

The action and the gauge invariant observables → invariants built out of the tensor and its dual.

## Tensor invariants as Colored Graphs

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$$T_{a^1 a^2 a^3} T_{b^1 b^2 b^3} T_{c^1 c^2 c^3} \bar T_{p^1 p^2 p^3} \bar T_{q^1 q^2 q^3} \bar T_{r^1 r^2 r^3}$$

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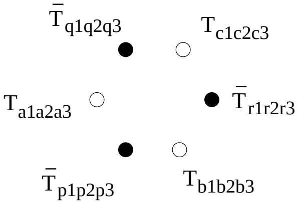
White (black) vertices for $T$ ($\bar T$).



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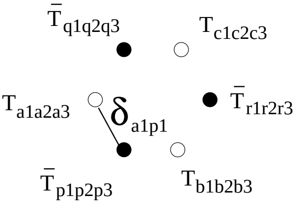

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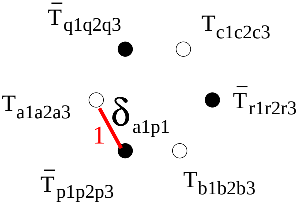
Edges for $\delta_{a^c q^c}$ colored by $c$, the position of the index.



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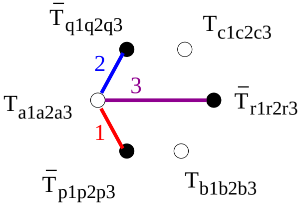
$$T_{a^1a^2a^3}\,T_{b^1b^2b^3}\,T_{c^1c^2c^3}\,\bar T_{p^1p^2p^3}\,\bar T_{q^1q^2q^3}\,\bar T_{r^1r^2r^3}$$

White (black) vertices for $T$ ($\bar T$).

Edges for $\delta_{a^cq^c}$ colored by $c$, the position of the index.



# Tensor invariants as Colored Graphs

$$T'_{b^1 \dots b^D} = \sum_a U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} T_{a^1 \dots a^D} \; , \quad \bar{T}'_{p^1 \dots p^D} = \sum_q \bar{U}^{(1)}_{p^1 q^1} \dots \bar{U}^{(D)}_{p^D q^D} \bar{T}_{q^1 \dots q^D}$$

Invariants ("traces") $\sum_{a^1, q^1} \delta_{a^1 q^1} \dots T_{a^1 \dots a^D} \bar{T}_{q^1 \dots q^D} \dots \to$ colored graphs $\mathcal{B}$.

$$D = 3 \; , \quad \sum \delta_{a^1 p^1} \delta_{a^2 q^2} \delta_{a^3 r^3} \quad \delta_{b^1 r^1} \delta_{b^2 p^2} \delta_{b^3 q^3} \quad \delta_{c^1 q^1} \delta_{c^2 r^2} \delta_{c^3 p^3}$$
$$T_{a^1 a^2 a^3} T_{b^1 b^2 b^3} T_{c^1 c^2 c^3} \bar{T}_{p^1 p^2 p^3} \bar{T}_{q^1 q^2 q^3} \bar{T}_{r^1 r^2 r^3}$$

White (black) vertices for $T$ ($\bar{T}$).

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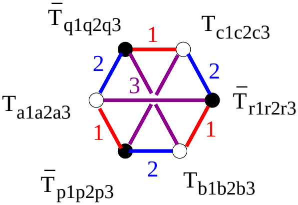

# Tensor invariants as Colored Graphs

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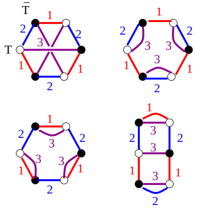
Invariants ("traces") $\sum_{a^1,q^1} \delta_{a^1 q^1} \dots T_{a^1\dots a^D}\,\bar T_{q^1\dots q^D}\dots \to$ colored graphs $\mathcal{B}$.

$$\mathrm{Tr}_{\mathcal{B}}(T,\bar T) = \sum \prod_v T_{a^1_v\dots a^D_v} \prod_{\bar v} \bar T_{q^1_{\bar v}\dots q^D_{\bar v}} \prod_{c=1}^{D} \prod_{e^c=(w,\bar w)} \delta_{a^c_w q^c_{\bar w}}$$

White (black) vertices for $T$ ($\bar T$).

Edges for $\delta_{a^c q^c}$ colored by $c$, the position of the index.



# Single trace models

The field → tensor:

$$T'_{b^1 \dots b^D} = \sum_a U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} T_{a^1 \dots a^D}$$

The action → "single trace" invariant:

$$S(T, \bar{T}) = \sum T_{b^1 \dots b^D} \bar{T}_{q^1 \dots q^D} \prod_{c=1}^{D} \delta_{b^c q^c} + \sum_{\substack{\text{connected graphs } \mathcal{B} \\ \text{with D colors}}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(T, \bar{T})$$

The partition function:

$$Z(t_{\mathcal{B}}) = \int [d\bar{T} dT] \, e^{-N^{D-1} S(T, \bar{T})}$$

The gauge invariant observables:

$$\mathrm{Tr}_{\mathcal{B}}(T, \bar{T})$$

Objective: $\ln Z$, $\langle \mathrm{Tr}_{\mathcal{B}_1}(T, \bar{T}) \dots \mathrm{Tr}_{\mathcal{B}_1}(T, \bar{T}) \rangle$

# Feynman expansion

$$S(T,\bar{T}) = \sum T_{b^1\dots b^D}\bar{T}_{q^1\dots q^D}\prod_{c=1}^{D}\delta_{b^c q^c} + \sum_{\substack{\text{connected graphs } \mathcal{B} \\ \text{with D colors}}} t_{\mathcal{B}} \mathrm{Tr}_{\mathcal{B}}(T,\bar{T}) \,,$$

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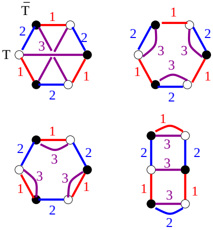
Feynman expansion:

- Taylor expand in $t_{\mathcal{B}} \to$ graphs with $D$ colors




$$Z(t_{\mathcal{B}}) = \sum \int_{T,\bar T} e^{-N^{D-1}\left(\sum T_{b^1\dots b^D}\, \bar T_{q^1\dots q^D} \prod_{c=1}^{D} \delta_{b^c q^c}\right)}$$
$$\mathrm{Tr}_{\mathcal{B}_1}(T,\bar T)\mathrm{Tr}_{\mathcal{B}_2}(T,\bar T)\dots$$

# Feynman expansion

$$S(T,\bar{T}) = \sum T_{b^1\dots b^D}\bar{T}_{q^1\dots q^D}\prod_{c=1}^{D}\delta_{b^c q^c} + \sum_{\substack{\text{connected graphs } \mathcal{B} \\ \text{with D colors}}} t_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(T,\bar{T}) \,,$$

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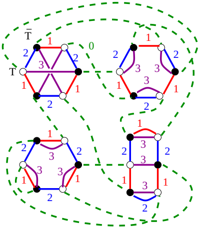
Feynman expansion:

- Taylor expand in $t_{\mathcal{B}}$ → graphs with $D$ colors
- compute the Gaussian integrals (Wick theorem) → graphs with $D+1$ colors




$$Z(t_{\mathcal{B}}) = \sum_{\text{graphs } \mathcal{G} \text{ with } D+1 \text{ colors}} A(\mathcal{G})$$

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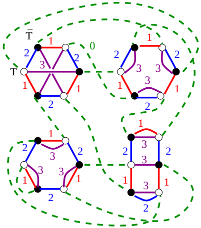
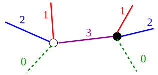



$$Z(t_{\mathcal{B}}) = \sum_{\text{graphs } \mathcal{G} \text{ with } D+1 \text{ colors}} A(\mathcal{G})$$

Each graph $\mathcal{G}$ is embedded in a $D$ dimensional space
(Poincaré dual to a triangulation)

# Colored graphs and vertex colored triangulations

White and black $D+1$ valent vertices connected by edges with colors $0, 1 \ldots D$.



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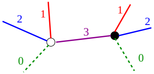
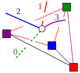
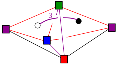



Vertex $\leftrightarrow$ $D$ simplex with colored vertices .




Edges $\leftrightarrow$ gluings along $D-1$ simplices respecting all the colorings



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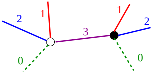


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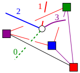


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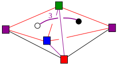
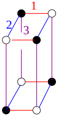
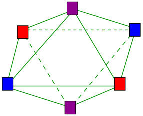



Invariants $\mathrm{Tr}_{\mathcal{B}}$: graphs with $D$ colors, $D-1$ dimensional boundary triangulations.



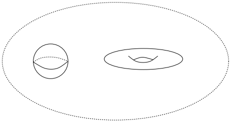
# Observables and expectations

Observables = invariants $\mathrm{Tr}_{\mathcal{B}}$ encoding boundary triangulations.
Expectations =

$$\left\langle \mathrm{Tr}_{\mathcal{B}_1} \mathrm{Tr}_{\mathcal{B}_2} \dots \mathrm{Tr}_{\mathcal{B}_q} \right\rangle = \frac{1}{Z(t_{\mathcal{B}})} \int [d\bar{T} dT] \, \mathrm{Tr}_{\mathcal{B}_1} \mathrm{Tr}_{\mathcal{B}_2} \dots \mathrm{Tr}_{\mathcal{B}_q} \, e^{-N^{D-1} S(T,\bar{T})}$$

correlations between boundary states given by sums over all bulk triangulations compatible with the boundary states

- $\left\langle \mathrm{Tr}_{\mathcal{B}_1} \mathrm{Tr}_{\mathcal{B}_2} \right\rangle_c = \left\langle \mathrm{Tr}_{\mathcal{B}_1} \mathrm{Tr}_{\mathcal{B}_2} \right\rangle - \left\langle \mathrm{Tr}_{\mathcal{B}_1} \right\rangle \left\langle \mathrm{Tr}_{\mathcal{B}_2} \right\rangle$: transition amplitude between the boundary states $\mathcal{B}_1$ and $\mathcal{B}_2$



- canonical measure over triangulations.

# The $1/N$ expansion for matrices

$$Z = \int [dMdM^{\dagger}] e^{-N\left\{ \mathrm{Tr}[MM^{\dagger}] - \sum_{p\geq 1} \frac{z^p}{p} \mathrm{Tr}[(MM^{\dagger})^p] \right\}}$$

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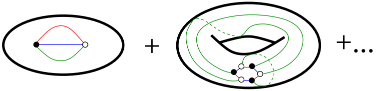
- perturbative expansion indexed by graphs embedded in surfaces

$$\ln Z = \sum_{\substack{\text{connected} \\ \text{2+1 colored graphs } \mathcal{G}}} A(\mathcal{G})$$



- the scaling with $N$ senses the topology (genus $g(\mathcal{G}) \geq 0$):

$$A(\mathcal{G}) \sim N^{2-2g(\mathcal{G})}$$

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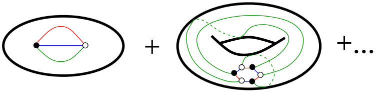

- the scaling with $N$ senses the topology (genus $g(\mathcal{G}) \geq 0$):

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- G. 't Hooft (1974) → reorganize the perturbative series in powers of $1/N$:

$$\frac{1}{N^2} \ln Z = \underbrace{\sum_{g \geq 0} \left(\frac{1}{N}\right)^{2g}}_{\text{expansion in topologies}} \underbrace{\sum_{\substack{\text{connected} \\ \text{2+1 colored graphs } \mathcal{G}}}^{g(\mathcal{G})=g} \frac{1}{p(\mathcal{G})} z^{p(\mathcal{G})}}_{\text{convergent sum at fixed topology}}$$

($p(\mathcal{G})$ ↙ number of white vertices)

# What replaces the genus?

The key in $D = 2$ is the Euler relation relating the number of vertices and faces (bi–colored cycles) of a graph

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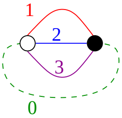
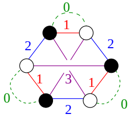
### Theorem

*For any $D$ and any connected $\mathcal{G}$, with $2p(\mathcal{G})$ vertices and $F(\mathcal{G})$ faces there exists a non negative integer $\omega(\mathcal{G})$ such that*

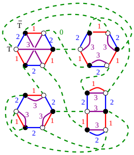
$$F(\mathcal{G}) = \frac{1}{2} D(D-1) p(\mathcal{G}) + D - \frac{2}{(D-1)!}\omega(\mathcal{G}) \,, \qquad \omega(\mathcal{G}) \geq 0 \,.$$




$\omega(\mathcal{G}) = 0$




$\omega(\mathcal{G}) = 4$



$\omega(\mathcal{G}) = 14$

# The $1/N$ expansion in $D \geq 3$

$$S(T,\bar{T}) = \sum T_{a^1\dots a^D} \bar{T}_{q^1\dots q^D} \prod_{i=1}^{D} \delta_{a^i q^i} - \sum_{\mathcal{B}} N^{-\frac{2}{(D-2)!}\omega(\mathcal{B})} \frac{z^{p(\mathcal{B})}}{p(\mathcal{B})} \mathrm{Tr}_{\mathcal{B}}(T,\bar{T})$$

$$Z = \int [d\bar{T} dT]\, e^{-N^{D-1} S(T,\bar{T})}$$

**Theorem**

*The free energy of this model admits the $1/N$ expansion:*

$$\frac{1}{N^D} \ln Z = \sum_{\omega \geq 0} N^{-\frac{2}{(D-1)!}\omega} \sum_{\substack{\mathcal{G} \text{ connected bipartite} \\ D+1 \text{ colored graphs}}}^{\omega(\mathcal{G})=\omega} \frac{1}{p(\mathcal{G})} z^{p(\mathcal{G})} .$$

Similar $1/N$ expansions exist for arbitrary observables.

# Topology and the $1/N$ expansion

For $D \geq 3$ there does not exist a unique topological invariant that discriminates the topologies.

The $1/N$ expansion can not be indexed by a topological invariant: $\omega(\mathcal{G})$ mixes topological and triangulation dependent information.

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- melonic graphs, $\omega(\mathcal{G}) = 0$, are spheres
- at any degree, only a finite number of topologies contribute.
- if a topology contributes at some degree, then it contributes an infinite number of triangulations

- a topology contributes to an infinite number of degrees.

## Leading order

Leading order $\omega(\mathcal{G}) = 0$:

$$\lim_{N\to\infty} \frac{1}{N^D} \ln Z = \sum_{\substack{\mathcal{G} \text{ connected bipartite} \\ D+1 \text{ colored graphs}}}^{\omega(\mathcal{G})=0} \frac{1}{p(\mathcal{G})} z^{p(\mathcal{G})} .$$

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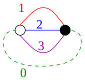
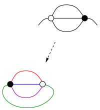
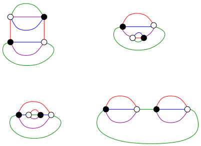
**Theorem**

*Graphs of degree 0 are melonic.*





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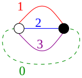


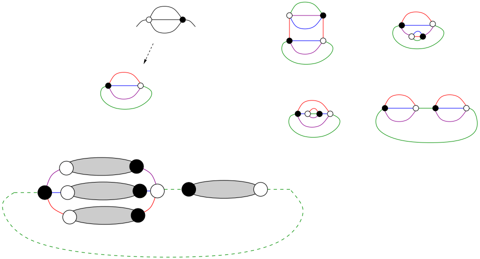

# Melonic theories

Whenever a random tensor is present, melonic graphs dominate.

In some cases: the self energy factors at leading order in terms of the two point function.

In general: random tensors are universal.

# A first example: The SYK model

Vector Majorana fermions $\chi_a(\tau)$, $q$–body interaction with quenched random couplings:

$$S = \frac{1}{2}\sum_a \int \chi_a \partial_\tau \chi_a + \sum_{a_1 \dots a_q} J_{a_1 \dots a_q} \int_\tau \chi_{a_1} \cdots \chi_{a_q} \ ,$$

$$d\nu(J) = e^{-\frac{N^{(q-1)}}{J^2} \sum_a (J_{a_1 \dots a_q})^2}$$

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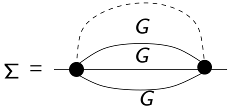
$$d\nu(J) = e^{-\frac{N^{(q-1)}}{J^2} \sum_a (J_{a_1 \dots a_q})^2}$$

$G(\tau_1, \tau_2) \to$ the disorder average of the thermal two point function:

$$G(\tau_1, \tau_2) = \int d\nu(J) \left[ \frac{\int [d\chi] \ e^{-S} \chi(\tau_1)\chi(\tau_2)}{\int [d\chi] \ e^{-S}} \right]$$

The disorder average (random tensor): melonic at leading order in $N$

$$\Sigma = J^2 G^{q-1}$$



# Melons in the SYK model

**Theorem**

*Gaussian distribution:* $\bar{T} \cdot T$

- *for any* $\mathcal{B}$ *with* $2p(\mathcal{B})$ *vertices, there exists a unique "order of convergence"* $\Omega(\mathcal{B}) \geq \omega(\mathcal{B}) \geq 0$ *and a unique* $R(\mathcal{B})$ *positive integer such that:*

$$\lim_{N\to\infty} N^{-1+\Omega(\mathcal{B})} \int [d\bar{T} dT] \, e^{-\frac{N^{D-1}}{C} \bar{T}\cdot T} \, Tr_{\mathcal{B}}(\bar{T}, T) = C^{p(\mathcal{B})} R(\mathcal{B})$$

- *furthermore, the scaling in N is the unique scaling such that:*
  - $\Omega(\mathcal{B}) \geq 0$ *for all* $\mathcal{B}$.
  - $\forall \mathcal{B}$ *there exists an infinite family of invariants* $\mathcal{B}'$ *such that* $\Omega(\mathcal{B}) = \Omega(\mathcal{B}')$.

- *finally, if* $\omega(\mathcal{B}) = 0$ *then* $\Omega(\mathcal{B}) = 0$ *and* $R(\mathcal{B}) = 1$.

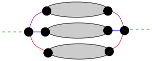
# Tensor SYK models

Promote everybody to tensor fields (Witten, 2016) $\chi_{a^1 \dots a^D}(\tau)$:

$$S = \frac{1}{2} \sum \int \chi \partial_\tau \chi + J \sum \int_\tau \underbrace{\chi \dots \chi}_{D+1} ,$$

$$Z = \int [d\chi] e^{-S} , \quad G = \frac{1}{Z} \int [d\chi] \, e^{-S} \chi\chi$$

Melonic at leading order in $1/N$:



$\Sigma = J^2 G^D$

# Why Tensor SYK?

- Eliminates the quenching without drowning the fermions (Witten)
- Number of fields does not proliferate for $N \to \infty$: as in QCD, it is the gauge group which grows (Witten)
- Solves the question of the singlets: only gauge invariant observables (Klebanov)
- Better controlled $1/N$ series (Klebanov)

# Universality

Theorem

*Gaussian distribution:* $\bar{T} \cdot T$

- *for any* $\mathcal{B}$ *with* $2p(\mathcal{B})$ *vertices, there exists a unique "degree of convergence"* $\Omega(\mathcal{B}) \geq \omega(\mathcal{B}) \geq 0$ *and a unique* $R(\mathcal{B})$ *positive integer such that:*

$$\lim_{N\to\infty} N^{-1+\Omega(\mathcal{B})} \int [d\bar{T}dT]\, e^{-\frac{N^{D-1}}{C}\bar{T}\cdot T}\, Tr_{\mathcal{B}}(\bar{T},T) = C^{p(\mathcal{B})} R(\mathcal{B})$$

- *furthermore, the scaling in* $N$ *is the unique scaling such that:*
  - $\Omega(\mathcal{B}) \geq 0$ *for all* $\mathcal{B}$.
  - $\forall \mathcal{B}$ *there exists an infinite family of invariants* $\mathcal{B}'$ *such that* $\Omega(\mathcal{B}) = \Omega(\mathcal{B}')$.

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Theorem

*Invariant distribution*: $S = \bar{T}\cdot T + \sum_{\mathcal{B} \text{ connected}} t_{\mathcal{B}} Tr_{\mathcal{B}}(\bar{T},T)$

- *in the large N limit*

$$\lim_{N\to\infty} N^{-1+\Omega(\mathcal{B})} \int \frac{[d\bar{T}dT]}{Z}\, e^{-N^{D-1}S(\bar{T},T)}\, Tr_{\mathcal{B}}(\bar{T},T) = G^{p(\mathcal{B})} R(\mathcal{B})$$

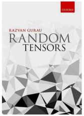
- *the covariance (two point function) G:*

$$1 - G - GV'_{\rm melo}(G) = 0\,, \qquad V_{\rm melo}(x) = \sum_{\mathcal{B}\ melonic} t_{\mathcal{B}} x^{p(\mathcal{B})}\,.$$

# The bottom line

Random tensors:

- canonical path integral formulation.
- built in scales (tensors of size $N^D$).
- new universal $1/N$ expansion, melons dominate.
- "interesting" $CFT_1$s, random geometries in arbitrary dimension, etc.




SYK model(s):

- $\imath\epsilon$ prescription preserving the symmetry, reproducing the conformal limit at $\epsilon \to 0$.