École thématique d'algèbre et topologie 2017

Sheaf theoretic methods in topology

Mixed Hodge Theory and Intersection Cohomology

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The cohomology of a smooth projective variety (or more generally a compact Kähler manifold), satisfies very nice properties, such as the Hodge decomposition, Hard Lefschetz or Poincaré duality. In this course we will discuss two main approaches to generalize these nice properties to the case of singular projective varieties: mixed Hodge theory and intersection cohomology. We will also discuss the interaction between these two approaches.

Lecture 1: Derived categories, sheaf cohomology and the 6 operations. The first lecture will contain most of the preliminaries on sheaf theory needed for both courses. I will briefly recall how to derive left-exact functors of abelian categories with enough injectives. I will then explain how compute sheaf cohomology via Godement's resolution, a canonical resolution by flasque sheaves. The functorial nature of Godement's resolution is particularly useful in mixed hodge thoery. Lastly, I will review Grothendieck's six operations on sheaves.
References [Ive86], [GM03], [God58].

Lecture 2: Generalities on the cohomology of complex algebraic varieties. I will review the Kähler package and explain the main properties of both intersection cohomology and mixed Hodge theory. In the case of projective varieties with isolated singularities, both theories allow for explicit simple descriptions at the same time that exhibit interesting phenomena. I will explain in detail the intersection cohomology and the mixed Hodge theory of some very simple examples.
References [GM80], [GM83], [KW06], [PS08], [Dur83].

Lecture 3: Mixed Hodge theory and the weight filtration as a new invariant. I will make precise some of the constructions of the previous talk. I will explain the main ideas behind Deligne's construction of mixed Hodge structures, using resolutions of singularities and the thoery of cohomological descent, and review some important properties of the weight filtration. I will also explain how the weight filtration can also be defined in other contexts, such as for the $\mathbb{Z}_2$-cohomology of real algebraic varieties or the cohomology with coefficients in an arbitrary ring of complex analytic spaces (endowed with a class of compactifications). In these cases, the weigh filtration is not part of a mixed hodge structure, but it still bive rise to an interesting algebraic (resp. analytic) invariant. I plan to compute several examples.
References [Del71], [Del74], [Tot02], [GNA02], [CG14], [MP11], [MP12].

Lecture 4: Mixed Hodge modules and the decomposition theorem. In this last lecture I will introduce Saito's theory of mixed Hodge modules in an axiomatic way, as a tool to extend Deligne's theory of mixed Hodge structures to the setting of intersection cohomology. I will also talk about purity of the weight filtration and the Decomposition Theorem, a very powerful tool for studying the topology of proper maps between algebraic varieties, relating their homological, Hodge-theoretic and arithmetic properties.
References [PS08], [dCM09], [dCM14a], [dCM14b].

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