Are there additional symmetries in Hopf algebras?

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Extracted from the book *Symmetry* by Hermann Weyl (Princeton University Press, 1952).

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Abstract

An involutory Hopf algebra is a Hopf algebra whose antipode squared equals the identity, $\mathcal{S}^2 = \mathrm{id}$. Few Hopf algebras are involutory but many more are almost involutory.

The antipode is not a morphism of Hopf algebras but it is an antimorphism. The identity map is an automorphism of Hopf algebras, hence it is tempting to substitute id $\mapsto \sigma$ where σ is an arbitray Hopf morphism and consider Hopf algebras whose antipode squared is the square of a Hopf homomorphism, $\mathcal{S}^2 = \sigma^2$. A map such as σ if it exists, is called a companion morphism.

If $\mathcal{S}$ has finite order, so does σ. A morphism of a given mathematical structure that is of finite order may be interpreted as a symmetry of the structure.

Hence, the companion morphism can be interpreted as an additional symmetry of the structure of *H*.

If the Hopf algebra *H* admits a companion morphism, we say that it is almost involutory (AI).

The purpose of this talk, is to define and consider the initial properties of almost involutory Hopf algebras. We prove that up to dimension 15 all Hopf algebras except a few types in dimensions eight and twelve are AI.

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The four operations: bialgebras

A bialgebra H is an algebra (H, m, u) endowed with a comultiplication and a counit, that are *dual* to multiplication and unit in the sense that as maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbb{K}$ satisfy a coassociativity and a counital condition respectively dual to associativity and unital condition. For example, in that duality the condition that the element 1 is a left unit corresponds to:

$$\mathrm{id} = (\varepsilon \otimes \mathrm{id})\Delta : H \to H \otimes H \to H \otimes \mathbb{K} = H,$$

and similarly for the other conditions.

The four operations satisfy certain compatibility restrictions that are the following:

1. The map $\Delta : H \to H \otimes H$ is a unital algebra map.
2. The map $\varepsilon : H \to \mathbb{K}$ is a unital algebra map (augmentation).

When the above operations are present we say that we have an associative and coassociative bialgebra.

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The antipode $\mathcal{S}$: Hopf algebras

One can think that a bialgebra corresponds to a monoid and that we need the inverse to make it a group. In order to do that, we have to introduce a new ingredient called the antipode –that will correspond in this analogy with the inverse–.

The antipode is a map $\mathcal{S} : H \to H$ that satisfies the following conditions:

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The positive antipode

Recall that a compact quantum group (CQG) –this concept is due to Woronowicz and was defined in order to give a *non commutative* version of the notion of compact topological group– is a Hopf algebra H over $\mathbb{C}$, equipped with a conjugate linear involution $x \mapsto x^*$ and an inner product, satisfying certain compatibility and positivity conditions. In general this concept is interesting in the infinite dimensional case.

Theorem (Woronowicz)

If H is a CQG, then $\mathcal{S}^2 : H \to H$ is a Hopf automorphism that is positive definite –with respect to the inner product defined in H.

In a joint paper with A. Abella and M. Haim (2009), we proved that the positive square root of $\mathcal{S}^2$–that always exists and is unique due to its positivity– is a Hopf automorphism (hence a companion automorphism) that we called the *positive antipode* : $\mathcal{S}_+$.

In 2003, Woronowicz, Masuda and Nakagami, proved that for a CQG the automorphism $\mathcal{S}^2$ is given by conjugation and we proved that $\mathcal{S}_+$ is also given by conjugation.

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The positive antipode as conjugation

That means that there is an algebra morphism $\gamma : H \to \mathbb{K}$, such that if $a \in H$ and $\Delta^2(a) = \sum a_i \otimes b_i \otimes c_i$, then:

$$\mathcal{S}^2(a) = \sum \gamma(a_i) b_i \gamma^{-1}(c_i).$$

We proved that for a certain augmentation map $\beta : H \to \mathbb{C}$, the companion automorphism is given as:

$$\mathcal{S}_+(a) = \sum \beta(a_i) b_i \beta^{-1}(c_i)$$

and from that it easily follows that $\beta^4 = \alpha$ where α is the modular function and that is a version of Radford's formula for CQG.
The conjugation formula presented above for $\mathcal{S}_+$ seems important because a similar conjugation formula for $\mathcal{S}$ cannot exist. This is because, conjugation produces morphisms of Hopf algebras and $\mathcal{S}$ is not.

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Order of $\mathcal{S}^2$ *odd*

In the case that the antipode is of finite order –for example if H is finite dimensional, we have that

$$(\mathcal{S}^2)^{2k-1} = \mathrm{id} \quad \Rightarrow \quad (\mathcal{S}^{2k})^2 = \mathcal{S}^2 \quad \Rightarrow \quad \sigma := \mathcal{S}^{2k}\,;\ \sigma^2 = \mathcal{S}^2,$$

then the Hopf algebra is almost involutive and $\mathcal{S}^{2k}$ is the companion automorphism we are looking for.

Thus, the problem of finding a companion morphism, i.e. a Hopf algebra morphism that is a square root of $\mathcal{S}^2$ is only difficult in the case that the order of the antipode is even.

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then the Hopf algebra is almost involutive and $\mathcal{S}^{2k}$ is the companion automorphism we are looking for.

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An elementary example: Sweedler's H_4.

As an algebra:

$$H_4 = \langle g, x : \; g^2 = 1, \; x^2 = 0, \; xg + gx = 0 \rangle.$$

As a coalgebra:

$$\Delta(g) = g \otimes g \quad , \quad \Delta(x) = x \otimes g + 1 \otimes x.$$

The antipode: $\mathcal{S}(g) = g$, $\mathcal{S}(x) = -xg$, $\mathcal{S}^2(g) = g$, $\mathcal{S}^2(x) = -x$.

The maps $\sigma_\pm : H_4 \to H_4$ defined as: $\sigma_\pm(g) = g$, $\sigma_\pm(x) = \pm ix$, and extended multiplicatively, are two different companion automorphisms for H_4.

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Recall that a Hopf algebra H is AI (almost involutive) if it admits a companion homomorphism $\sigma^2 = \mathcal{S}^2$.

Assume that H and K are AI Hopf algebras, it is clear that:

1. $H \otimes K$ is an AI Hopf algebra.
2. If H is finite dimensional and AI, H^* is also an AI Hopf algebra.
3. If H is finite dimensional and AI so is $D(H)$, the Drinfel'd double.
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Items 1,2, and 4 are clear. Later we deal with the third.
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A matched pair of Hopf algebras is a pair of Hopf algebras (A, H) and a pair of maps:

$$H \stackrel{\triangleleft}{\leftarrow} H \otimes A \stackrel{\triangleright}{\rightarrow} A,$$
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subject to certain compatibility conditions:

- The maps $\triangleleft, \triangleright$ are coalgebra maps and actions of A and H respectively.
- The actions $\triangleleft, \triangleright$ satisfy the following compatibility conditions:

$$x \triangleright ab = \sum (x_1 \triangleright a_1)\big((x_2 \triangleleft a_2) \triangleright b\big), \quad x \triangleright 1 = \varepsilon(x)1,$$
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In this context, the following theorem is well known:

First general results III: the bicrossed product

Theorem

In the above situation, endowing $A \otimes H$ with the tensor product coproduct and with the product and antipode:

$$(a \otimes x)(b \otimes y) = \sum a(x_1 \triangleright b_1) \otimes (x_2 \triangleleft b_2)y,$$

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we obtain a Hopf algebra $A \bowtie H$: the bicrossed product of A and H.

Moreover, the bicrossed product of AI Hopf algebras is an AI Hopf algebra provided that the companion morphisms of the factors satisfy the following compatibility conditions:

$$\sigma_A(x \triangleright a) = \sigma_H(x) \triangleright \sigma_A(a) \quad , \quad \sigma_H(x \triangleleft a) = \sigma_H(x) \triangleleft \sigma_A(a)$$

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$$\sigma_{A \bowtie H}(a \otimes x) = \sigma_A(a) \otimes \sigma_H(x).$$

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If H is a finite dimensional AI Hopf algebra so is $D(H)$.

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$$(a \otimes x)(b \otimes y) = \sum a(x_1 \triangleright b_1) \otimes (x_2 \triangleleft b_2)y,$$

$$\mathcal{S}(a \otimes x) = \sum (\mathcal{S}(x_2) \triangleright \mathcal{S}(a_2)) \otimes (\mathcal{S}(x_1) \triangleleft \mathcal{S}(a_1)),$$

we obtain a Hopf algebra $A \bowtie H$: the bicrossed product of A and H.

Moreover, the bicrossed product of AI Hopf algebras is an AI Hopf algebra provided that the companion morphisms of the factors satisfy the following compatibility conditions:

$$\sigma_A(x \triangleright a) = \sigma_H(x) \triangleright \sigma_A(a) \quad , \quad \sigma_H(x \triangleleft a) = \sigma_H(x) \triangleleft \sigma_A(a)$$

In this case

$$\sigma_{A \bowtie H}(a \otimes x) = \sigma_A(a) \otimes \sigma_H(x).$$

Corollary

If H is a finite dimensional AI Hopf algebra so is $D(H)$.

Trivial extensions of Hopf algebras

We assume that the Hopf algebra H has a decomposition:

$$H = K \oplus M,$$

and that:

1. $K \subseteq H$ is a sub Hopf algebra.
2. M is a K–bimodule and a K–bicomodule, i.e:
   2.1 $KM + MK \subseteq M$;
   2.2 $\Delta(M) \subseteq K \otimes M + M \otimes K$;
3. The extension of K by M is trivial, in other words, $M^2 = 0$.
4. M is invariant by $\mathcal{S}$.

In this situation one has that $\varepsilon(M) = 0$.

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Trivial extensions II

We add the following conditions:

1. K is almost involutive with companion morphism σ_K.
2. There is a linear map $\sigma_M : M \to M$, with the property that $\sigma_M^2 = \mathcal{S}^2|_M$ and such that:
   2.1 $\sigma_M(hm) = \sigma_K(h)\sigma_M(m)$, $\sigma_M(mh) = \sigma_M(m)\sigma_K(h)$.
   2.2 If $\Delta(m) = \sum h_i \otimes m_i + n_i \otimes k_i \in K \otimes M + M \otimes K$, then $\Delta(\sigma_M(m)) = \sum \sigma_K(h_i) \otimes \sigma_M(m_i) + \sigma_M(n_i) \otimes \sigma_K(k_i)$.

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M & \xrightarrow{\quad \sigma_M \quad} & M \\
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$$\begin{CD} M @>{\sigma_M}>> M \\ @V{\Delta}VV @VV{\Delta}V \\ K \otimes M + M \otimes K @>>{\sigma_K \otimes \sigma_M + \sigma_M \otimes \sigma_K}> K \otimes M + M \otimes K \end{CD}$$

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Trivial extensions III

Theorem: The map $\sigma(h+m) = \sigma_K(h) + \sigma_M(m)$ is a companion automorphism for H.

Proof: Indeed, it is clear that $\sigma^2 = \mathcal{S}^2$.

1. σ multiplicative,

   $\sigma((h+m)(k+n)) = \sigma(hk+hn+mk) = \sigma_K(hk) + \sigma_M(hn+mk) = \sigma_K(h)\sigma_K(k) + \sigma_K(h)\sigma_M(n) + \sigma_M(m)\sigma_K(k) = (\sigma_K(h) + \sigma_M(m))(\sigma_K(k) + \sigma_M(n)) = \sigma(h+m)\sigma(k+n)$.

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A situation that happens frequently –for example in the case of pointed Hopf algebras– is that we have a trivial extension and the subHopf algebra coincides with the 1–eigenspace of $\mathcal{S}^2$, i.e.

$$K = \left\{x \in H : \mathcal{S}^2(x) = x\right\}.$$

In that situation, K involutive by definition.

Sweedler's Hopf algebra as an example.

$$H_4 = \{1, g, x, gx\}_{\mathbb{K}} = \{1, g\}_{\mathbb{K}} \oplus \{x, gx\}_{\mathbb{K}} = K \oplus M.$$

In this situation $\sigma_K = \mathrm{id}$ and $\sigma_M = i\,\mathrm{id}$ satisfy all the required properties.

An eight dimensional example.

$$\langle g, h, x : g^2 = 1,\ h^2 = 1,\ x^2 = 0,\ gx{+}xg = 0,\ hx{+}xh = 0,\ gh{-}hg = 0\rangle.$$

Where g and h are group-like elements and x is a $(g, 1)$-primitive element

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Consider the eight dimensional Hopf algebra given as:

$$A''_{C_4} = \langle g, x : \; g^4 = 1, \; x^2 = g^2 - 1, \; gx + xg = 0 \rangle \,,$$

where g is a group-like element and x is a $(g, 1)$-primitive element.

In this situation the antipode squared has order two and its eigenspaces are:

$$E_1 = \{1, g, g^2, g^3\}_{\mathbb{K}} \,, \; E_{-1} = \{x, gx, g^2x, g^3x\}_{\mathbb{K}}.$$

If σ exists, it leaves invariant $E_{\pm 1}$. As g is the modular element, we have that $\sigma(g) = g$, as the integral $\ell = (1 + g + g^2 + g^3)x$ is mapped into an integral, then $\sigma(\ell) = p\ell$ with $p^2 = -1$. Hence, $\sigma(x) = px$.

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$$A''_{C_4} = \langle g, x : \; g^4 = 1, \; x^2 = g^2 - 1, \; gx + xg = 0 \rangle \,,$$

where g is a group-like element and x is a $(g, 1)$-primitive element.

In this situation the antipode squared has order two and its eigenspaces are:

$$E_1 = \{1, g, g^2, g^3\}_{\mathbb{K}} \,, \; E_{-1} = \{x, gx, g^2x, g^3x\}_{\mathbb{K}}.$$

If σ exists, it leaves invariant $E_{\pm 1}$. As g is the modular element, we have that $\sigma(g) = g$, as the integral $\ell = (1 + g + g^2 + g^3)x$ is mapped into an integral, then $\sigma(\ell) = p\ell$ with $p^2 = -1$. Hence, $\sigma(x) = px$.

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Future work

1. Can the pointed AI Hopf algebras be classified?
2. Work out the categorical viewpoint or is the representation theory of an AI Hopf algebra, special? How is it special?
3. Classify the (co) quasitriangular AI Hopf algebras.
4. Recalling that in the case of CQG, the companion automorphism is given by conjugation, does the same happens –under certain restrictions– for finite dimensional AI Hopf algebras?

Gracias a los organizadores y a los participantes.

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