How Teissier mixed multiplicities

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Bernard Teissier’s wisdom

“Better a house without roof than a house without view.”
Hunza saying

(starting aphorism of [Bernard Teissier: Introduction to equisingularity problems. Proc. of Symp. in Pure Maths. 29 (1975), 593-632.] )

“What I wish to discuss here today is [...] how to classify singularities in characteristic zero, in fact [...] in the complex domain. [...] The most substantial contributions here were made by differential topologists rather than by algebraic geometers. [...] the purely algebraic approach while still in its infancy, seems to be the most natural approach to the subject, for it is doubtful whether singularities of complex-analytic varieties are purely topological or even differential-geometric phenomena.”
“The basic question is the following: what shall we mean by saying that the two singularities $P, P'$ are equivalent? The relation of equivalence which we are trying to spell out and which we shall designate by the term “equisingularity” should formalize our vague and not very intuitive idea of singularities of the same type, of the same degree of complexity. One thing is clear: it must be an equivalence relation which is much weaker than an analytical isomorphism.”
“Any definition of equisingularity should imply equimultiplicity, at the very least. Thus our first question is the following:

A. Does topological equisingularity of $V_r$ and $V'_r$ at $P$ and $P'$ imply that $e(V_r, P) = e(V'_r, P')$?”
“Let me now take a new tack which promises a better wind. Instead of dealing with a pair of hypersurfaces, let us consider analytic families of hypersurfaces \( V_r \), all having a singular point at the origin and depending on a set of parameters. […]

D. Does topological equisingularity imply differential equisingularity ?”

Here “differential equisingularity” is to be understood in terms of Whitney conditions, that is, one has “nice” behavior of limits of tangent planes and of chords.
The basic counterexample to Whitney’s conditions

Let us come back to Zariski’s paper:

“One may cite at this point also the following result, due to Hironaka [Heisuke Hironaka: Normal cones in analytic Whitney stratifications. Publ. Math. IHES 36 (1969), 127-138.]:

Differential equisingularity of $V_n$ at $P$, along $W$, implies equimultiplicity of $V_n$ at $P$, along $W$.”
A kind of converse

But Hironaka had started even before to think about the relation between differential geometric and algebraic equisingularity. For instance, in [H. Hironaka : *Equivalence and deformations of singularities*. Preprint, 1964.] he stated with a sketch of proof the following theorem:

**Theorem**

Consider a family of germs of complex analytic varieties over a smooth base $Y$, with total space $X$. Identify $Y$ with the subvariety of $X$ of base points of the germs. Blow up the product of the defining ideal of $Y$ and of the relative jacobian ideal of the family. If the resulting family of analytic spaces is equidimensional, then the pair $(X_{smooth}, Y)$ satisfies Whitney’s conditions along $Y$.

A complete proof was provided by [Jean-Paul Speder : *Éclatements jacobiens et conditions de Whitney*. In *Singularités à Cargèse*, Astérisque 7-8 (1973), 47-66.]
In [B. Teissier: *Cycles évanouissants et conditions de Whitney*. CRAS 276 (1973), 1051-1054.], the abstract states:

“We give a numerical necessary and sufficient condition for a complex analytic hypersurface to satisfy Whitney’s conditions along its singular locus (in a neighborhood of a smooth point of it) in terms of the number of vanishing cycles of the fibers of a retraction of the hypersurface onto its singular locus and of the sections of those fibers by generic planes of various dimensions.”

Here:

“number of vanishing cycles” = Milnor number
The sequence $\mu^*$

“[...] we attach to a germ of hypersurface $(X_0, x_0) \subset (\mathbb{C}^{n+1}, 0)$” with isolated singularity a decreasing sequences of integers

$$\mu^*_x(X_0) = (\mu^{(n+1)}_x(X_0), ..., \mu^{(i)}_x(X_0), ..., \mu^{(0)}_x(X_0))$$

where $\mu^{(i)}_x(X_0)$ is the number of vanishing cycles of the intersection of $(X_0, x_0)$ with a general $i$-plane of $(\mathbb{C}^{n+1}, 0)$.”

Teissier proved:

**Theorem**

Let $G : (X, x) \mapsto (D, 0)$ be a germ of deformation of a hypersurface with isolated singularity, endowed with a section $D$ such that $X \setminus D$ is smooth above $D$. If $\mu^*$ is constant in this family, then the pair of strata $(X \setminus D, D)$ satisfies Whitney’s conditions at any point of $D$.

The converse was proved in [Joël Briançon, Jean-Paul Speder: Les conditions de Whitney impliquent “$\mu^*$ constant”. Ann. Inst. Fourier (Grenoble) 26 (1976), no. 2, 153-163.]
Note that one has $\mu_{x_0}^{(1)}(X_0) = e_{x_0}(X_0) - 1$.

Teissier’s strategy to prove that topological equisingularity implies equimultiplicity was to “go downstairs” along the sequence $\mu^*$:

**Conjecture**

If $(X_0, x_0)$ and $(X_1, x_1)$ have the same topological type, one has:

$$\mu_{x_0}^*(X_0) = \mu_{x_1}^*(X_1).$$

In [J. Briançon, J.-P. Speder : *La trivialité topologique n’implique pas les conditions de Whitney*. C. R. Acad. Sci. Paris Sér. A-B 280 (1975), no. 6, A365-A367.] this conjecture was shown to be false, even in a 1-parameter family.

**Zariski’s conjecture is still open!**
When working on [Cargèse 1973], Teissier tried to prove his conjecture by searching the way in which the knowledge of \( \mu^{(n+1)} \) constrains the other \( \mu^{(i)} \)'s. For instance, he proved that:

**Proposition**

\[
\mu^{(n+1)} \geq \mu^{(1)} \cdot \mu^{(n)} \quad \text{(therefore} \quad \mu^{(i+1)} \geq \mu^{(1)} \cdot \mu^{(i)}, \ \forall \ 1 \leq i \leq n.)
\]

**Proof.** Let \( \Gamma \) be the curve defined by \( \frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_n} = 0 \). If \( z_0 = 0 \) is general enough relative to \( X_0 := f^{-1}(0) \), then:

\[
\mu^{(n+1)} + \mu^{(n)} = (X_0 \cdot \Gamma)_0 \geq m_{x_0}(X_0) \cdot m_{x_0}(\Gamma) = (\mu^{(1)} + 1)\mu^{(n)}.
\]

The curve \( \Gamma \) is here a computational tool. It was in Cargèse that Teissier learnt from Lê Dũng Tráng that it was known under the name of **polar curve**. At that time, Thom was a promoter of its use in singularity theory. **Polar varieties** in general were to become one of the great loves of Bernard!
In [Cargèse 1973] Teissier asked also:

**Question**  
Is it always true that:

\[
\frac{\mu^{(n+1)}}{\mu^{(n)}} \geq \frac{\mu^{(n)}}{\mu^{(n-1)}} \geq \cdots \geq \frac{\mu^{(1)}}{\mu^{(0)}}
\]

Here \(\mu^{(0)} := 1\). Note that those inequalities are equivalent to the fact that:

\[
\log \mu^{(i)} \leq \frac{1}{2} \left( \log \mu^{(i-1)} + \log \mu^{(i+1)} \right)
\]

which explains the expression "log-convexity".
Teissier himself answered this question affirmatively several years later. In fact he proved more general inequalities, for **mixed multiplicities** of an arbitrary pair of primary ideals in a regular local ring.

He was led to introduce this notion inspired by the suggestion of Hironaka to consider the function $K : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by:

$$K(r, s) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n+1}, 0}}{\mathcal{M}^r \cdot j(f)^s}.$$ 

where $\mathcal{M} = (z_0, ..., z_n)$ is the maximal ideal of the local ring $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$ and $j(f) = (\frac{\partial f}{\partial z_0}, ..., \frac{\partial f}{\partial z_n})$ is the jacobian ideal of $f$. Recall that Hironaka blew-up their product!

**Theorem**

Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be two primary ideals in a noetherian local ring $\mathcal{O}$. Then the function $H : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ defined by:

$$H(a_1, a_2) := \text{length} \frac{\mathcal{O}}{\mathfrak{n}_1^{a_1} \mathfrak{n}_2^{a_2}}$$

is polynomial when $a_1$ and $a_2$ are big enough, and its degree is equal to the Krull dimension of $\mathcal{O}$.

This extends to two ideals the approach of Pierre Samuel (1951) for defining the multiplicity $e(\mathfrak{n})$ of one primary ideal.
The mixed multiplicities of two ideals

In [Cargèse 1973], Teissier extended this theorem (with the help of Jean-Jacques Risler) to any finite number of ideals, in a version relative to an arbitrary \( \mathcal{O} \)-module, and he defined some “symbols”, which he called later mixed multiplicities. Let us look at the way he defined them in the case of two ideals, the \( \mathcal{O} \)-module being for simplicity \( \mathcal{O} \) itself:

**Definition**

Let \( \overline{H}(a_1, a_2) \) be the homogeneous part of highest degree \( d = \dim \mathcal{O} \) of the polynomial \( H \). The **symbols** \( \left[ n_1^{[k_1]}, n_2^{[k_2]} \right] \) are defined by:

\[
\overline{H}(a_1, a_2) = \sum_{k_1+k_2=d} \frac{1}{k_1! \cdot k_2!} \left[ n_1^{[k_1]}, n_2^{[k_2]} \right] a_1^{k_1} \cdot a_2^{k_2}.
\]
Mixed multiplicities are classical multiplicities

Teissier proved in [Cargèse 1973] :

**Proposition**

Let $k_1 + k_2 = d = \text{dim } O$. The symbol $\left[ n_1^{[k_1]}, n_2^{[k_2]} \right]$ is equal to the (Samuel) multiplicity of an ideal generated by $k_1$ general elements of $n_1$ and $k_2$ general elements of $n_2$.

Therefore, those symbols are classical multiplicities, but of ideals obtained by mixing elements of $n_1$ and $n_2$. This explains the name he chose later for them.

In fact he introduced this name only when he became conscious of a deep analogy with Minkowski’s mixed volumes of convex bodies.
The previous interpretation of mixed multiplicities allowed him to prove:

**Proposition**

Let $f \in \mathcal{M} \subset \mathcal{O}_{\mathbb{C}^{n+1},0}$ be a function with isolated singularity. For all $i \in \{0, \ldots, n+1\}$, one has:

$$\mu^{(i)} = [\mathcal{M}^{[n+1-i]}, j(f)^{[i]}].$$

The basic idea is that $n+1-i$ general elements of $\mathcal{M}$ define a general $i$-plane through the origin.

The subtle point is that one has to compare the restriction of the jacobian ideal $j(f)$ to such an $i$-plane and the jacobian ideal of the restriction of $f$. This is the point where integral closures of ideals enter the game. This notion was very important in subsequent work of Bernard on the local structure of complex spaces.
Let us come back to an arbitrary pair of primary ideals in a noetherian local ring. Teissier proved the following “symbolic binomial formula”:

**Proposition**

\[ e(n_1n_2) = \sum_{i=0}^{d} \binom{d}{i} [n_1^{[d-i]}, n_2^{[i]}]. \]
This suggested him:

**Question**

Is it always true that $[n_1^{d-i}, n_2^i]^d \leq e(n_1)^{d-i} \cdot e(n_2)^i$? This would imply the "**Minkowski-type inequality**":

$$(e(n_1n_2))^{1/d} \leq (e(n_1))^{1/d} + (e(n_2))^{1/d}.$$  

Teissier saw this as an analog of **Minkowski’s inequality**:  

$$\left(\sum_k (x_k + y_k)^d\right)^{1/d} \leq \left(\sum_k x_k^d\right)^{1/d} + \left(\sum_k y_k^d\right)^{1/d}.$$
The log-convexity of mixed multiplicities

In [B. Teissier : Sur une inégalité à la Minkowski pour les multiplicités. Appendix to David Eisenbud and Harold Levine : An algebraic formula for the degree of a \( C^\infty \)-map germ. Ann. Math. 106 (1977), 19-44 (38-44).] he proved the stronger inequalities:

**Theorem**

Assume that \( \mathcal{O} \) is a reduced Cohen-Macaulay algebra over an algebraically closed field of characteristic zero. Let \( d \) be its dimension. Then, by denoting \( e(i) := [n_1^{[d-i]}, n_2^{[i]}] \), one has:

\[
\frac{e(d)}{e(d-1)} \geq \frac{e(d-1)}{e(d-2)} \geq \cdots \geq \frac{e(1)}{e(0)}.
\]

The log-convexity inequalities for \( \mu^* \) are special cases!
The principle of the proof

- By **successive hyperplane sections** using general elements of both ideals, **reduce to the case of a surface**.
- There, **lift the ideals to a resolution**.
- The inequalities are a consequence of the fact that **the intersection form of this resolution is negative definite**.
A corollary for finite map germs

Corollary

Let $F : (\mathbb{C}^n, 0) \mapsto (\mathbb{C}^n, 0)$ be a finite map germ. If $\Gamma$ is the preimage of a generic line through the origin in the target space, then:

$$m_0(\Gamma) \leq (\deg F)^{1 - \frac{1}{n}}.$$

In fact this was a question of Eisenbud from 1975, which stimulated Teissier to come back to his question. Eisenbud and Levine deduced from the previous corollary that:

Corollary

Let $F : (\mathbb{R}^n, 0) \mapsto (\mathbb{R}^n, 0)$ be a real analytic map germ such that its complexification has finite degree $\deg_{\mathbb{C}} F$. Then $F$ has also finite degree $\deg_{\mathbb{R}} F$ and:

$$\deg_{\mathbb{R}}(F) \leq (\deg_{\mathbb{C}} F)^{1 - \frac{1}{n}}.$$
We saw before an argument based on the fact that the intersection multiplicity of a curve and a hypersurface on a smooth space is greater or equal to the product of the multiplicities of the intersected germs. Teissier proved the following avatar on a possibly singular surface:

**Corollary**

Let \((S, 0)\) be a germ of normal surface and \(C_1, C_2\) be two germs of effective Weil divisors on it. Then:

\[
(C_1, C_2)_0 \geq \frac{m_0(C_1) \cdot m_0(C_2)}{m_0(S)}.
\]

Here one uses Mumford’s notion of rational intersection number of two Weil divisors.
The characterization of equality

In [B. Teissier: *On a Minkowski-type inequality for multiplicities - II*, In C.P. Ramanujam: a Tribute, Springer-Verlag, 1978.] is proved:

**Theorem**

One has equality in the Minkowski-type inequality if and only if there exist positive integers \(a, b\) such that \(n_1^a\) and \(n_2^b\) have the same integral closure.

In [David Rees and Rodney Sharp: *On a theorem of B. Teissier on multiplicities of ideals in local rings*. J. London Math. Soc. (2) **18** (1978), 449-463.], the log-convexity inequalities and the characterization of the equality case are extended to more general noetherian local rings.
Teissier deduced from his characterization of equality:

**Corollary**

Let $F : (\mathbb{R}^2, 0) \mapsto (\mathbb{R}^2, 0)$ be a germ of real-analytic map with finite complex degree $\deg_{\mathbb{C}} f$. Then $f$ can be continuously deformed with constant real and complex degrees to a germ of holomorphic mapping $(\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$, if and only if:

$$\deg_{\mathbb{R}} f = \sqrt{\deg_{\mathbb{C}} f}.$$ 

At the end of his paper, he asks to find invariants which allow to decide when it is possible to deform two real analytic map germs $F_{1,2} : (\mathbb{R}^n, 0) \mapsto (\mathbb{R}^n, 0)$ one into the other with constant real and complex degrees.

This question is still open.
Arrived at this point, I could continue:

- either by describing how he discovered global “Khovanski-Teissier inequalities” and “Bonnesen-type inequalities” for positive enough line bundles on projective varieties;

- or by describing the way he characterized the Whitney conditions at a point of a smooth stratum in a complex analytic variety using “polar multiplicities”, and related work done with Lê on limits of tangent spaces.

But my time is over. I offer you no roof, but a view:

Teissier’s papers!
Happy birthday Bernard ! ! !

Keep great energy and intuition in your explorations of singular landscapes!