How Teissier mixed multiplicities

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"**Better a house without roof than a house without view.**
Hunza saying"

(starting aphorism of [Bernard Teissier : **Introduction to equisingularity problems.** Proc. of Symp. in Pure Maths. **29** (1975), 593-632.])

Let us look at some quotations from [Oscar Zariski : **Some open questions in the theory of singularities.** Bulletin A.M.S. **77** No. 4 (1971), 481-491.]

> *"What I wish to discuss here today is [...] how to* classify *singularities* in characteristic zero, *in fact [...]* in the complex domain*. [...] The most substantial contributions here were made by differential topologists rather than by algebraic geometers. [...] the purely* **algebraic** *approach while still in its infancy, seems to be the most natural approach to the subject, for it is doubtful whether singularities of complex-analytic varieties are purely* **topological** *or even* **differential-geometric** *phenomena."*

Zariski's basic question on equivalence of singularities

"The basic question is the following : **what shall we mean by saying that the two singularities P, P' are equivalent ?** *The relation of equivalence which we are trying to spell out and which we shall designate by the term "***equisingularity***" should formalize our vague and not very intuitive idea of singularities of the same type, of the same degree of complexity. One thing is clear : it must be an equivalence relation which is much weaker than an analytical isomorphism."*

Zariski's question on multiplicities

"Any definition of equisingularity should imply **equimultiplicity***, at the very least. Thus our first question is the following :*

A. Does topological equisingularity of V_r and V_r' at P and P' imply that $e(V_r, P) = e(V_r', P')$?"

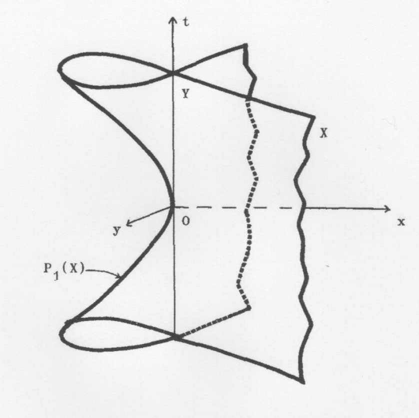
Zariski's questions about equisingular families

"Let me now take a new tack which promises a better wind. Instead of dealing with a pair *of hypersurfaces, let us consider* analytic families *of hypersurfaces V_r, all having a singular point at the origin and depending on a set of parameters. [...]*

D. Does topological equisingularity imply differential equisingularity ?"

Here "differential equisingularity" is to be understood in terms of **Whitney conditions**, that is, one has "nice" behavior of limits of tangent planes and of chords.

The basic counterexample to Whitney's conditions



[B. Teissier : **Variétés polaires II. Multiplicités polaires, sections planes et conditions de Whitney.** In **Algebraic geometry** (La Rábida, 1981), 314-491, Lecture Notes in Math. **961**, Springer, Berlin, 1982.]

Hironaka's equimultiplicity theorem

Let us come back to Zariski's paper :

> *"One may cite at this point also the following result, due to Hironaka [Heisuke Hironaka :* **Normal cones in analytic Whitney stratifications.** *Publ. Math. IHES* **36** *(1969), 127-138.] :*
>
> **Differential equisingularity of V_n at P, along W, implies equimultiplicity of V_n at P, along W."**

A kind of converse

But Hironaka had started even before to think about the relation between differential geometric and algebraic equisingularity. For instance, in [H. Hironaka : **Equivalence and deformations of singularities.** Preprint, 1964.] he stated with a sketch of proof the following theorem :

Theorem

Consider a family of germs of complex analytic varieties over a smooth base Y, with total space X. Identify Y with the subvariety of X of base points of the germs. Blow up the product of the **defining ideal of Y** and of the **relative jacobian ideal of the family**. If the resulting family of analytic spaces is equidimensional, then the pair (X_{smooth}, Y) satisfies Whitney's conditions along Y.

A complete proof was provided by [Jean-Paul Speder : **Éclatements jacobiens et conditions de Whitney.** In **Singularités à Cargèse**, Astérisque **7-8** (1973), 47-66.]

Teissier's CRAS announcement

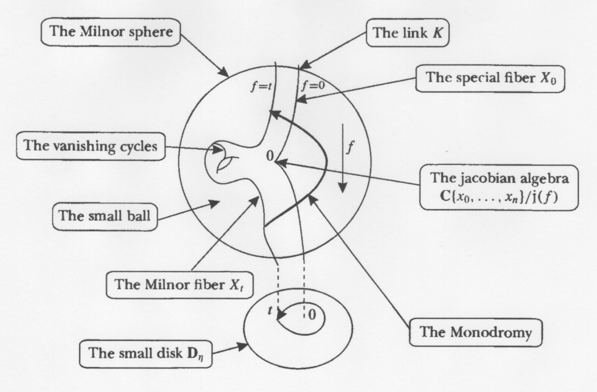
In [B. Teissier : **Cycles évanouissants et conditions de Whitney.** CRAS **276** (1973), 1051-1054.], the abstract states :

> *"We give a numerical necessary and sufficient condition for a complex analytic hypersurface to satisfy Whitney's conditions along its singular locus (in a neighborhood of a smooth point of it) in terms of the* **number of vanishing cycles** *of the fibers of a retraction of the hypersurface onto its singular locus* **and of the sections of those fibers by generic planes of various dimensions***."*

Here :

"number of vanishing cycles" $=$ **Milnor number**

Milnor's gift to singularists



[B. Teissier : **A bouquet of bouquets for a birthday.** In **Topological methods in modern mathematics.** A Symposium in honor of John Milnor's sixtieth birthday. Publish or Perish, 1993, 93-122.]

"[...] we attach to a germ of hypersurface $(X_0, x_0) \subset (\mathbb{C}^{n+1}, 0)$" with isolated singularity a decreasing sequences of integers $\mu^*_{x_0}(X_0) = (\mu^{(n+1)}_{x_0}(X_0), ..., \mu^{(i)}_{x_0}(X_0), ..., \mu^{(0)}_{x_0}(X_0))$ where $\mu^{(i)}_{x_0}(X_0)$ is the number of vanishing cycles of the intersection of (X_0, x_0) with a general i-plane of $(\mathbb{C}^{n+1}, 0)$."

[B. Teissier : **Cycles évanescents, sections planes et conditions de Whitney.** In **Singularités à Cargèse** 285-362. Astérisque, Nos. 7 et 8, Soc. Math. France, Paris, 1973.]

The numerical criterion for differential equisingularity

Teissier proved :

Theorem

Let $G : (X, x) \mapsto (\mathbb{D}, 0)$ be a germ of deformation of a hypersurface with isolated singularity, endowed with a section $\mathbb{D}$ such that $X \setminus \mathbb{D}$ is smooth above $\mathbb{D}$. If μ^* is constant in this family, then the pair of strata $(X \setminus \mathbb{D}, \mathbb{D})$ satisfies Whitney's conditions at any point of $\mathbb{D}$.

The converse was proved in [Joël Briançon, Jean-Paul Speder : **Les conditions de Whitney impliquent "μ^* constant"**. Ann. Inst. Fourier (Grenoble) **26** (1976), no. 2, 153-163.]

Note that one has $\mu^{(1)}_{x_0}(X_0) = e_{x_0}(X_0) - 1$.
Teissier's strategy to prove that topological equisingularity implies equimultiplicity was to "go downstairs" along the sequence μ^* :

Conjecture

If (X_0, x_0) and (X_1, x_1) have the same topological type, one has : $\mu^*_{x_0}(X_0) = \mu^*_{x_1}(X_1)$."

In [J. Briançon, J.-P. Speder : **La trivialité topologique n'implique pas les conditions de Whitney.** C. R. Acad. Sci. Paris Sér. A-B **280** (1975), no. 6, A365-A367.] this conjecture was shown to be false, even in a 1-parameter family.

Zariski's conjecture is still open !

Teissier was looking for inequalities

When working on [Cargèse 1973], Teissier tried to prove his conjecture by searching the way in which the knowledge of $\mu^{(n+1)}$ constrains the other $\mu^{(i)}$'s. For instance, he proved that :

Proposition

$\mu^{(n+1)} \geq \mu^{(1)} \cdot \mu^{(n)}$ (therefore $\mu^{(i+1)} \geq \mu^{(1)} \cdot \mu^{(i)}, \forall\, 1 \leq i \leq n$.)

Proof. Let Γ be the curve defined by $\frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_n} = 0$. If $z_0 = 0$ is general enough relative to $X_0 := f^{-1}(0)$, then :

$$\mu^{(n+1)} + \mu^{(n)} = (X_0 \cdot \Gamma)_0 \geq m_{x_0}(X_0) \cdot m_{x_0}(\Gamma) = (\mu^{(1)} + 1)\mu^{(n)}. \square$$

The curve Γ is here a computational tool. It was in Cargèse that Teissier learnt from Lê Dũng Tráng that it was known under the name of **polar curve**. At that time, Thom was a promoter of its use in singularity theory. **Polar varieties** in general were to become one of the great loves of Bernard !

Is μ^* log-convex ?

In [Cargèse 1973] Teissier asked also :

Question

Is it always true that :

$$\frac{\mu^{(n+1)}}{\mu^{(n)}} \geq \frac{\mu^{(n)}}{\mu^{(n-1)}} \geq \cdots \geq \frac{\mu^{(1)}}{\mu^{(0)}} \ ?$$

Here $\mu^{(0)} := 1$. Note that those inequalities are equivalent to the fact that :

$$\log \mu^{(i)} \leq \frac{1}{2} \left(\log \mu^{(i-1)} + \log \mu^{(i+1)} \right)$$

which explains the expression "log-convexity".

Hironaka's suggestion

Teissier himself answered this question affirmatively several years later. In fact he proved more general inequalities, for **mixed multiplicities** of an arbitrary pair of primary ideals in a regular local ring.

He was led to introduce this notion inspired by the suggestion of Hironaka to consider the function $K : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ defined by :

$$\mathbf{K(r, s)} := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{\mathcal{M}^r \cdot j(f)^s}.$$

where $\mathcal{M} = (z_0, ..., z_n)$ is the maximal ideal of the local ring $\mathcal{O}_{\mathbb{C}^{n+1},0}$ and $\mathbf{j(f)} = (\frac{\partial f}{\partial z_0}, ..., \frac{\partial f}{\partial z_n})$ is the jacobian ideal of f. Recall that Hironaka blew-up their product !

Bhattacharya's theorem

One has the following theorem proved in [Phani Bhushan Bhattacharya : **The Hilbert function of two ideals.** Math. Proc. Cambridge Philo. Society **53** (3) (1957), 568-575] :

Theorem

Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be two primary ideals in a noetherian local ring $\mathcal{O}$. Then the function $H : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ defined by :

$$H(a_1, a_2) := \text{length} \frac{\mathcal{O}}{\mathfrak{n}_1^{a_1} \mathfrak{n}_2^{a_2}}$$

is polynomial when a_1 and a_2 are big enough, and its degree is equal to the Krull dimension of $\mathcal{O}$.

This extends to **two** ideals the approach of Pierre Samuel (1951) for defining the **multiplicity** $e(\mathfrak{n})$ of **one** primary ideal.

The mixed multiplicities of two ideals

In [Cargèse 1973], Teissier extended this theorem (with the help of Jean-Jacques Risler) to any finite number of ideals, in a version relative to an arbitrary $\mathcal{O}$-module, and he defined some "**symbols**", which he called later **mixed multiplicities**. Let us look at the way he defined them in the case of two ideals, the $\mathcal{O}$-module being for simplicity $\mathcal{O}$ itself :

Definition

Let $\overline{H}(a_1, a_2)$ be the homogeneous part of highest degree $d = \dim \mathcal{O}$ of the polynomial H. The **symbols** $\left[\mathfrak{n}_1^{[k_1]}, \mathfrak{n}_2^{[k_2]}\right]$ are defined by :

$$\overline{H}(a_1, a_2) = \sum_{k_1+k_2=d} \frac{1}{k_1! \cdot k_2!} \left[\mathfrak{n}_1^{[k_1]}, \mathfrak{n}_2^{[k_2]}\right] a_1^{k_1} \cdot a_2^{k_2}.$$

Mixed multiplicities are classical multiplicities

Teissier proved in [Cargèse 1973] :

Proposition

Let $k_1 + k_2 = d = \dim \mathcal{O}$. The symbol $\left[\mathfrak{n}_1^{[k_1]}, \mathfrak{n}_2^{[k_2]}\right]$ is equal to the (Samuel) multiplicity of an ideal generated by k_1 general elements of n_1 and k_2 general elements of n_2.

Therefore, those symbols are classical multiplicities, but of ideals obtained by **mixing** elements of $\mathfrak{n}_1$ and $\mathfrak{n}_2$. This explains the name he chose later for them.

In fact he introduced this name only when he became conscious of a deep analogy with Minkowski's **mixed volumes** of convex bodies.

μ^* is a collection of mixed multiplicities

The previous interpretation of mixed multiplicities allowed him to prove :

Proposition

Let $f \in \mathcal{M} \subset \mathcal{O}_{\mathbb{C}^{n+1},0}$ be a function with isolated singularity. For all $i \in \{0, ..., n+1\}$, one has :

$$\mu^{(i)} = [\mathcal{M}^{[n+1-i]}, j(f)^{[i]}].$$

The basic idea is that $n+1-i$ general elements of $\mathcal{M}$ define a general i-plane through the origin.

The subtle point is that one has to compare **the restriction of the jacobian ideal $j(f)$ to such an i-plane** and **the jacobian ideal of the restriction of f**. This is the point where **integral closures of ideals** enter the game. This notion was very important in subsequent work of Bernard on the local structure of complex spaces.

The multiplicity of a product of ideals

Let us come back to an arbitrary pair of primary ideals in a noetherian local ring. Teissier proved the following "symbolic binomial formula" :

Proposition

$$e(\mathfrak{n}_1\mathfrak{n}_2) = \sum_{i=0}^{d} \binom{d}{i} [\mathfrak{n}_1^{[d-i]}, \mathfrak{n}_2^{[i]}].$$

A conjectural Minkowski-type inequality

This suggested him :

Question

Is it always true that $[\mathfrak{n}_1^{[d-i]}, \mathfrak{n}_2^{[i]}]^d \leq e(\mathfrak{n}_1)^{d-i} \cdot e(\mathfrak{n}_2)^i$? This would imply the "**Minkowski-type inequality**" :

$$(e(\mathfrak{n}_1\mathfrak{n}_2))^{1/d} \leq (e(\mathfrak{n}_1))^{1/d} + (e(\mathfrak{n}_2))^{1/d} .$$

Teissier saw this as an analog of **Minkowski's inequality** :

$$\left(\sum_k (x_k + y_k)^d\right)^{1/d} \leq \left(\sum_k x_k^d\right)^{1/d} + \left(\sum_k y_k^d\right)^{1/d}$$

In [B. Teissier : **Sur une inégalité à la Minkowski pour les multiplicités**. Appendix to David Eisenbud and Harold Levine : **An algebraic formula for the degree of a C^∞-map germ.** Ann. Math. **106** (1977), 19-44 (38-44).] he proved the stronger inequalities :

Theorem

Assume that $\mathcal{O}$ is a reduced Cohen-Macaulay algebra over an algebraically closed field of characteristic zero. Let d be its dimension. Then, by denoting $e^{(i)} :=: [\mathfrak{n}_1^{[d-i]}, \mathfrak{n}_2^{[i]}]$, one has :

$$\frac{e^{(d)}}{e^{(d-1)}} \geq \frac{e^{(d-1)}}{e^{(d-2)}} \geq \cdots \geq \frac{e^{(1)}}{e^{(0)}}.$$

The log-convexity inequalities for μ^* are special cases !

The principle of the proof

- By **successive hyperplane sections** using general elements of both ideals, **reduce to the case of a surface**.
- There, **lift the ideals to a resolution**.
- The inequalities are a consequence of the fact that **the intersection form of this resolution is negative definite**.

A corollary for finite map germs

Corollary

Let $F : (\mathbb{C}^n, 0) \mapsto (\mathbb{C}^n, 0)$ be a finite map germ. If Γ is the preimage of a generic line through the origin in the target space, then :

$$m_0(\Gamma) \leq (\deg F)^{1-\frac{1}{n}}.$$

In fact this was a question of Eisenbud from 1975, which stimulated Teissier to come back to his question. Eisenbud and Levine deduced from the previous corollary that :

Corollary

Let $F : (\mathbb{R}^n, 0) \mapsto (\mathbb{R}^n, 0)$ be a real analytic map germ such that its complexification has finite degree $\deg_{\mathbb{C}} F$. Then F has also finite degree $\deg_{\mathbb{R}} F$ and :

$$\deg_{\mathbb{R}}(F) \leq (\deg_{\mathbb{C}} F)^{1-\frac{1}{n}}.$$

Another corollary for intersection numbers on surfaces

We saw before an argument based on the fact that the intersection multiplicity of a curve and a hypersurface **on a smooth space** is greater or equal to the product of the multiplicities of the intersected germs. Teissier proved the following avatar **on a possibly singular surface** :

Corollary

Let $(S, 0)$ be a germ of normal surface and C_1, C_2 be two germs of effective Weil divisors on it. Then :

$$(C_1, C_2)_0 \geq \frac{m_0(C_1) \cdot m_0(C_2)}{m_0(S)}.$$

Here one uses Mumford's notion of **rational** intersection number of two Weil divisors.

The characterization of equality

In [B. Teissier : **On a Minkowski-type inequality for multiplicities - II**, In **C.P. Ramanujam : a Tribute**, Springer-Verlag, 1978.] is proved :

Theorem

One has equality in the Minkowski-type inequality if and only if there exist positive integers a, b such that $\mathfrak{n}_1^a$ and $\mathfrak{n}_2^b$ have the same integral closure.

In [David Rees and Rodney Sharp : **On a theorem of B. Teissier on multiplicities of ideals in local rings.** J. London Math. Soc. (2) **18** (1978), 449-463.], the log-convexity inequalities and the characterization of the equality case are extended to more general noetherian local rings.

A corollary in dimension two

Teissier deduced from his characterization of equality :

Corollary

Let $F : (\mathbb{R}^2, 0) \mapsto (\mathbb{R}^2, 0)$ be a germ of real-analytic map with finite complex degree $\deg_{\mathbb{C}} f$. Then f can be continuously deformed with constant real and complex degrees to a germ of holomorphic mapping $(\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$, if and only if :

$$\deg_{\mathbb{R}} f = \sqrt{\deg_{\mathbb{C}} f}.$$

At the end of his paper, he asks **to find invariants which allow to decide when it is possible to deform two real analytic map germs $F_{1,2} : (\mathbb{R}^n, 0) \mapsto (\mathbb{R}^n, 0)$ one into the other with constant real and complex degrees**.

This question is still open.

Possible continuations

Arrived at this point, I could continue :

- either by describing how he discovered global "**Khovanski-Teissier inequalities**" and "**Bonnesen-type inequalities**" for positive enough line bundles on projective varieties ;
- or by describing the way he characterized the Whitney conditions at a point of a smooth stratum in a complex analytic variety using "**polar multiplicities**", and related work done with Lê on **limits of tangent spaces**.

But my time is over. I offer you no roof, but a view :

Teissier's papers !

Happy birthday Bernard ! ! !

Keep great energy and intuition in your explorations of singular landscapes !