Local Monomialization of Analytic Maps

Steven Dale Cutkosky

At Aussois, in honor of
Bernard Teissier

Let $\varphi : (Y, b) \to (X, a)$ be a germ of a morphism of complex analytic spaces. If X and Y are varieties, then φ is regular if $\varphi(Y)$ contains an open subset of X (in the Euclidean topology).

Let $\mathrm{reg}(Y)$ be the nonsingular locus of Y.

φ is regular if and only if the open set

$$U = \{p \in \mathrm{reg}(Y) \mid \mathrm{rank}\ d\varphi_p = \dim X\}$$

is nonempty.

Gabrielov: If φ is not regular, it is possible for $\mathcal{O}^{\mathrm{an}}_{X,a} \to \mathcal{O}^{\mathrm{an}}_{Y,b}$ to be injective, but $\hat{\mathcal{O}}^{\mathrm{an}}_{X,a} \to \hat{\mathcal{O}}^{\mathrm{an}}_{Y,b}$ to be not injective. (The Zariski subspace theorem fails for analytic maps).

A local blow up of an analytic space X is a blow up $\pi : X' \to U$ where U is an open subset of X (in the Euclidean topology) and π is the blow up of a closed analytic subspace of U.

(An inclusion of an open subset U of X into X is a special case.)

Hironaka: An étoile e over an analytic space X is a subcategory of the category of finite sequences of local blowups over X which satisfies certain good properties.

To each $\pi : X' \to X \in e$ there is an associated point $e_{X'} \in X'$.

Given a factorization

$$X' = X_n \stackrel{\pi_n}{\to} X_{n-1} \to \cdots \to X_1 \stackrel{\pi_1}{\to} X$$

by local blow ups, we have $\pi_i(e_{X_i}) = e_{X_{i-1}}$ for all i.

An étoile e is a generalization of a valuation of an algebraic function field.

We can associate to an étoile over a nonsingular analytic space X a valuation $\nu = \nu_e$ on a giant field extension of $\mathrm{QF}(\mathcal{O}^{\mathrm{an}}_{X,e_X})$. The valuation ring V_e is constructed by taking the union of $\mathcal{O}^{\mathrm{an}}_{X',e_{X'}}$ where $X' \to X \in e$ is a sequence of local blow ups of nonsingular sub varieties. We have

$$\text{rank } \nu \leq \text{ratrank } \nu \leq \dim X.$$

Composite valuations can be very badly behaved!

Example: (C) There exists an étoile e on $Y_0 = \mathbb{C}^4$ such that the valuation ring V_e has a proper prime ideal Q such that there exists an infinite chain of local blow ups (of a point in Y_m if m is even and of a nonsingular surface if m is odd)

$$\cdots \to Y_m \to \cdots \to Y_1 \to Y_0$$

with $Y_m \to Y_0 \in e$ for all m, such that the center of Q on Y_m has dimension 3 if m is even and the center of Q on Y_m has dimension 2 if m is odd.

The construction begins with an example from the paper "Platificateur local en geometrie analytique et aplatissement local" in Singularités à Cargèse, by Hironaka, Lejeune-Jalabert and Teissier of an analytic map of a surface to a 3-fold such that the image is Zariski dense, but the Zariski closure of the image becomes a surface after blowing up an appropriate point in the 3-fold.

La Voûte Étoilée

$\mathcal{E}_X$ = set of all étoiles over X with a topology making $P_X : \mathcal{E}_X \rightarrow X$, $e \mapsto e_X$ continuous.

Theorem: (Hironaka) p_X is proper.

This theorem is a generalization of Zariski's theorem on the quasi compactness of the Zariski Riemann manifold.

Theorem: (C) Suppose that $\varphi : Y \to X$ is a morphism of reduced complex analytic spaces and $e \in \mathcal{E}_Y$. Then there exists a commutative diagram

$$\begin{array}{ccc} Y_1 & \stackrel{\varphi_1}{\to} & X_1 \\ \beta \downarrow & & \downarrow \alpha \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

such that X_1 and Y_1 are smooth, $\beta \in e$, φ_1 is regular, β is a finite product of local blow ups of nonsingular analytic sub varieties, α is a finite product of blowups of nonsingular analytic sub varieties and inclusions of proper analytic sub varieties.

The proof is a little delicate, but the essential ingredient is the local flattening theorem of Hironaka, Lejeune-Jalabert and Teissier in “Platificateur local en geometrie analytique et aplatissement local” or the later proof by Hironaka in his notes “Introduction to real-analytic sets and real-analytic maps” given at the University of Pisa.

Definition: Suppose that $\varphi : Y \to X$ is a regular morphism of complex or real analytic manifolds and $p \in Y$. We will say that φ is monomial at p if there exist regular parameters $x_1, \ldots, x_m$ in $\mathcal{O}^{\mathrm{an}}_{X,\varphi(p)}$ and $y_1, \ldots, y_n$ in $\mathcal{O}^{\mathrm{an}}_{Y,p}$ and $c_{ij} \in \mathbb{N}$ such that

$$x_i = \prod_{j=1}^{n} y_j^{c_{ij}} \text{ for } 1 \leq i \leq m.$$

(Since φ is regular we have $\mathrm{rank}(c_{ij}) = m$).

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We will say that $\varphi : Y \to X$ is monomial on Y if there exists an open cover of Y by open subsets U_k which are isomorphic to open subsets of $\mathbb{C}^n$ (or $\mathbb{R}^n$) and an open cover of X by open subsets V_k which are isomorphic to open subsets of $\mathbb{C}^m$ (or $\mathbb{R}^m$) such that $\varphi(U_i) \subset V_k$ for all k, and there exist $c_{ij}(k) \in \mathbb{N}$ such that

$$\varphi^*(x_i) = \prod_{j=1}^{n} y_j^{c_{ij}(k)} \text{ for } 1 \leq i \leq m$$

where x_i and y_j are the respective coordinates on $\mathbb{C}^m$ and $\mathbb{C}^n$ (or $\mathbb{R}^m$ and $\mathbb{R}^n$).

We will say that $y_1 y_2 \cdots y_n = 0$ is a local toroidal structure O on U_k.

Theorem: (C) Suppose that $\varphi : Y \to X$ is a regular morphism of complex analytic manifolds, E_Y is a simple normal crossings divisor on Y and e is an étoile over Y. Then there exists a commutative diagram

$$\begin{array}{ccc} Y_e & \stackrel{\varphi_e}{\to} & X_e \\ \pi_e \downarrow & & \downarrow \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

of regular analytic morphisms such that the vertical arrows are products of local blow ups of nonsingular analytic subvarieties, $Y_e \to Y \in e$ and φ_e is a monomial morphism for a toroidal structure O_e on Y_e. Further, we have that $\pi_e^*(E_Y)$ is an effective divisor supported on O_e and the restriction of π_e to $Y_e \setminus O_e$ is an open immersion.

Corollary: (C) Suppose that $\varphi : Y \to X$ is a regular morphism of complex (or real) analytic manifolds and $p \in Y$. Then there exists a finite number of commutative diagrams

$$\begin{array}{ccc} Y_i & \stackrel{\varphi_i}{\to} & X_i \\ \pi_i \downarrow & & \downarrow \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

of analytic morphisms, for $1 \le i \le t$, such that the vertical arrows are finite products of local blow ups of nonsingular analytic sub varieties, each φ_i is a monomial morphism of complex (or real) manifolds, and there exist compact subsets K_i of Y_i such that $\cup_{i=1}^{t} \pi_i(K_i)$ is a compact neighborhood of p in Y.

The Voûte Étoilée does not exist for real analytic spaces (as an étoile does not always have a real center). The corollary is proven in the real case using Hironaka's theory of complexification of real analytic spaces and maps.

Even "resolution of singularities" of real analytic spaces does not exist in a natural way.

The Whitney Umbrella

The singular variety $x^2 - zy^2 = 0$ is resolved (over $\mathbb{C}$) by blowing up the (complex) line $x = y = 0$.

Over $\mathbb{R}$, this singularity is the union of the negative z axis and a surface Y.

The real part W of the blow up of the line in the Whitney Umbrella maps into Y $(z \geq 0)$.

Hironaka: The resolution of singularities of the (real) Whitney Umbrella is the disjoint union of W and the (real) z-axis.

Theorem: Suppose that $\varphi : Y \to X$ is a morphism of reduced complex analytic spaces and $e \in \mathcal{E}_Y$ is an étoile over Y. Then there exists a commutative diagram of complex analytic morphisms

$$\begin{array}{ccc} Y_e & \stackrel{\varphi_e}{\to} & X_e \\ \beta \downarrow & & \downarrow \alpha \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

such that $\beta \in e$ is a finite product of local blow ups of nonsingular analytic sub varieties, α is a finite product of local blow ups of nonsingular analytic sub varieties and inclusions of analytic sub varieties, Y_e and X_e are nonsingular analytic spaces and φ_e is a regular, monomial analytic morphism for a toroidal structure O_e on Y_e.

If A is a given analytic subspace of Y, we further have that $\mathcal{I}_A\mathcal{O}_{Y_e} = \mathcal{O}_{Y_e}(-G)$ where $\mathcal{I}_A$ is the ideal sheaf in $\mathcal{O}_Y^{\rm an}$ of the analytic subspace A of Y, G is an effective divisor which is supported on O_e, and having the further condition that the restriction $(Y_e \setminus O_e) \to Y$ is an open embedding.

Corollary: (C) Suppose that $\varphi : Y \to X$ is a morphism of reduced complex analytic spaces and $p \in Y$. Then there exists a finite number of commutative diagrams

$$\begin{array}{ccc} Y_i & \stackrel{\varphi_i}{\to} & X_i \\ \beta_i \downarrow & & \downarrow \alpha_i \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

of analytic morphisms, for $1 \le i \le t$, such that X_i and Y_i are smooth, φ_i is monomial for a toroidal structure O_i on Y_i, β_i is a finite product of local blow ups of nonsingular analytic sub varieties, α_i is a finite product of blowups of nonsingular analytic sub varieties and inclusions of proper analytic sub varieties, and there exist compact subsets K_i of Y_i such that $\cup_{i=1}^t \beta_i(K_i)$ is a compact neighborhood of p in Y.

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Some comments on the proof of local monomialization along an étoile

Let e be an étoile over X, ν_e a valuation associated to e with valuation ring V_e.

Suppose $\tilde{X} \to X \in e$ and $x_1, \ldots, x_n$ is a regular system of parameters in $\mathcal{O}_{\tilde{X}, e_{\tilde{X}}}$. Suppose $\overline{X} \to \tilde{X}$ is such that $\overline{X} \to \tilde{X} \to X \in e$. The germ $\mathcal{O}^{\rm an}_{\tilde{X}, e_{\tilde{X}}} \to \mathcal{O}^{\rm an}_{\overline{X}, e_{\overline{X}}}$ is a general monomial transform (GMT) along e if $\mathcal{O}_{\overline{X}, e_{\overline{X}}}$ has regular parameters $\overline{x}_1, \ldots, \overline{x}_n$ such that there exists an $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \in \mathbb{N}$ and $\mathrm{Det}(a_{ij}) = \pm 1$ such that

$$x_i = \prod_{j=1}^{n} (\overline{x}_j + \alpha_j)^{a_{ij}}$$

for $1 \le j \le n$ and $\alpha_j \in \mathbb{C}$.

Let $S = \{i_1, \ldots, i_m\}$. A GMT is in the variables $x_{i_1}, \ldots, x_{i_m}$ if

$$x_i = \prod_{j \in S} (\overline{x}_j + \alpha_j)^{a_{ij}} \text{ for } i \in S \text{ and}$$

$$x_i = \overline{x}_i \text{ for } i \notin S$$

A GMT is said to be monomial if all α_j are zero.

Definition: The variables $x_1, \ldots, x_s$ are independent if every GMT along e in $x_1, \ldots, x_s$ is monomial..

Lemma: When performing a GMT of Perron type (in $x_1, \ldots, x_s$ which are independent or $x_1, \ldots, x_s, x_l$ with x_l dependent on $x_1, \ldots, x_s$) the condition that $x_1, \ldots, x_s$ are independent is preserved.

We inductively construct commutative diagrams

$$\begin{array}{ccc} \tilde{Y} & \rightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \rightarrow & X \end{array} \tag{1}$$

where the vertical arrows are products of blow ups of nonsingular sub varieties.

There are regular parameters $x_1, \ldots, x_m$ in $\mathcal{O}^{\rm an}_{\tilde X, e_{\tilde X}}$ and $y_1, \ldots, y_n$ in $\mathcal{O}^{\rm an}_{\tilde Y, e_{\tilde Y}}$ such that $y_1, \ldots, y_s$ are independent but $y_1, \ldots, y_s, y_i$ are dependent for all i with $s+1 \le i \le n$, $x_1, \ldots, x_r$ are independent, and there is an expression for some l,

$$
\begin{array}{lll}
x_1 & = & y_1^{c_{11}} \cdots y_s^{c_{1s}} \\
 & \vdots & \\
x_r & = & y_1^{v_{r1}} \cdots y_s^{c_{rs}} \\
x_{r+1} & = & y_{s+1} \\
 & \vdots & \\
x_{r+l} & = & y_{s+l}
\end{array}
\tag{2}
$$

We will say that the variables (x, y) are prepared of type (s, r, l).

We have $(s_1, r_1, l_1) > (s, r, l)$ if $s_1 \geq s$, $r_1 \geq r$, $r_1 + l_1 \geq r + l$ and one of these inequalities is strict.

Theorem: Given a diagram (1) and (2), there exists a sequence of GMT of Perron type which are also products of blow ups of nonsingular sub varieties, and some changes of variables in independent variables of Tschirnhaus type, giving a diagram (1) and (2) with

$$(s_1, r_1, l_1) > (s, r, l).$$

The Algebraic Case

Theorem: (C) Suppose that k is a field of characteristic zero and $\varphi : Y \to X$ is a dominant morphism of varieties over k. Suppose that ν is a valuation of $k(Y)$ dominating Y. Then there exists a commutative diagram

$$\begin{array}{ccc} Y_1 & \stackrel{\varphi_1}{\to} & X_1 \\ \downarrow & & \downarrow \\ Y & \stackrel{\varphi}{\to} & X \end{array}$$

such that the vertical arrows are products of monodial transforms (blow ups of nonsingular sub varieties) and φ_1 is monomial at the center of ν.

Such a diagram is called a local monomialization of φ along ν.

Suppose K^*/K is a finite field extension. ν^* is a valuation of $K^*$and $\nu = \nu^*|K$.

$$e = e(\nu^*/\nu) = [\Gamma_{\nu^*} : \Gamma_\nu]$$

$$f = f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu].$$

If ν^* is the unique extension of ν to K^*, then

$$[K^* : K] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}$$

where p is the characteristic of V_{ν^*}/m_{ν^*}.

$\delta(\nu^*/\nu)$ is the defect of ν^* over ν.

$\delta = 0$ if V_{ν^*}/m_{ν^*} has characteristic zero or if V_{ν^*} is discrete.

Theorem: (Piltant, C) Suppose that $\varphi : Y \rightarrow X$ is a morphism of two dimensional excellent surfaces, such that $k(Y)$ is finite separable over $k(X)$, ν^* is a valuation dominating $k(Y)$, $\nu = \nu^*|k(X)$ and the defect $\delta(\nu^*/\nu) = 0$. Then there exists a local monomialization of φ along ν^*.

Example: (C) There exist two dimensional examples (in any characteristic $p > 0$) and of a valuation ν such that local monomialization does not hold along ν.

The valuation has defect $2 > 0$.