# Rees algebras of codimension three Gorenstein ideals

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Problem B $\Longrightarrow$ Problem A

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The Rees algebra is the bi-homogeneous coordinate ring of the graph of $\Phi$.

In fact, the diagram

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corresponds to the diagram

$$\begin{array}{ccc} \mathcal{R}(I) & \twoheadleftarrow & k[x_1, \ldots, x_d, y_1, \ldots, y_n] =: S \\ \bigcup & & \bigcup \\ A(X) & \twoheadleftarrow & k[y_1, \ldots, y_n] =: T \\ f_j t & \longleftarrow & y_j \end{array}$$

where all rings are standard bi-graded with $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$.

The defining ideal of the Rees ring is the bi-homogeneous ideal $\mathcal{J}$ with $\mathcal{R}(I) \cong S/\mathcal{J}$. Restricting to the component of $\underline{x}$-degree zero, one obtains a presentation $A(X) \cong T/I(X)$.

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Problems:

- Find generators of the defining ideal $\mathcal{J}$ of $\mathcal{R}(I)$.
- Find bounds for the bi-degrees of these generators.
- When is $I$ of fiber type?

There is a vast body of work on these questions, in commutative algebra, elimination theory, algebraic geometry, and applied mathematics. Some of the articles relevant to this talk are:

Herzog-Simis-Vasconcelos (1982), Vasconcelos (1991), Geramita-Gimigliano (1991), Morey (1996), Morey-Ulrich (1996), Johnson (1997), Jouanolou (1997), Busé-Chardin (2005), Eisenbud-Huneke-Ulrich (2006), Song-Chen-Goldman (2006), Hong-Simis-Vasconcelos (2008), Cox-Hoffman-Wang (2008), Busé-Chardin-Jouanolou (2009), Busé-Chardin-Simis (2010), Kustin-Polini-Ulrich (2011), Cortadellas-D'Andrea (2013), Nguyen (2014)

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$$\mathrm{Sym}(I(\delta)) = S/\mathcal{L}$$

where $\mathcal{L}$ is generated by the entries of the vector $[y_1 \ldots y_n] \cdot \varphi$.

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where $\mathcal{L}$ is generated by the entries of the vector $[y_1 \ldots y_n] \cdot \varphi$.

Hence: To describe the defining ideal $\mathcal{J}$ of $\mathcal{R}(I)$ it suffices to determine $\mathcal{A} = \mathcal{J}/\mathcal{L}$.

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I \text{ is of linear type} &\iff \mathcal{R}(I) \cong \mathrm{Sym}(I(\delta)) \\
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I \text{ is of fiber type} &\iff \mathcal{A} = I(X) \cdot \mathrm{Sym}(I(\delta)) \\
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**Theorem [Herzog-Simis-Vasconcelos]**

$I$ is of linear type if

- $I$ is SCM: the Koszul homology of $I$ is Cohen-Macaulay, and
- $I$ is $G_\infty$: $\mu(I_p) \leq \operatorname{codim} p \quad \forall p \in V(I)$.

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$$\begin{aligned} \text{SCM} \quad &\overset{\text{[Huneke]}}{\Longleftarrow} \quad I \text{ is in the linkage class of a complete intersection} \\ &\Longleftarrow \quad \operatorname{codim} I = 2 \text{ with } R/I \text{ is CM, or} \\ &\qquad\; \operatorname{codim} I = 3 \text{ with } R/I \text{ Gorenstein} \end{aligned}$$

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$$G_\infty \quad \Longrightarrow \quad \mu(I) \leq d$$

# Zeroth local cohomology

Write $\mathfrak{m} = (x_1, \ldots, x_d) \subset R$ and $\mathcal{S} = \mathrm{Sym}(I(\delta))$

Recall $\mathcal{A} = \ker(\mathcal{S} \twoheadrightarrow \mathcal{R}(I))$

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$$\begin{aligned} \mathcal{A} = H^0_{\mathfrak{m}}(\mathcal{S}) \quad &\Longleftrightarrow \quad \mathcal{J} = \mathcal{L} : \mathfrak{m}^{\infty} \\ &\Longleftrightarrow \quad I_p \text{ is of linear type } \forall p \neq \mathfrak{m} \\ &\Longleftarrow \quad I \text{ is SCM and } G_d : \mu(I_p) \leq \operatorname{codim} p \quad \forall p \neq \mathfrak{m} \end{aligned}$$

# Constructing elements in $\mathcal{L} : \mathfrak{m}^\infty$

Assume $I$ is linearly presented:

$$\bigoplus R(-1) \xrightarrow{\varphi} R^n \xrightarrow{[f_1 \ldots f_n]} I(\delta) \longrightarrow 0$$

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$$[y_1 \ldots y_n] \cdot \varphi = [x_1 \ldots x_d] \cdot B$$

with $B$ a matrix with linear entries in $T = k[y_1, \ldots, y_n]$.

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Hence: $I_d(B) \subset \mathcal{L} : \mathfrak{m}$ in $S = k[\underline{x}, \underline{y}]$

The defining ideal $\mathcal{J}$ of $\mathcal{R}(I)$ is said to have the expected form if $\mathcal{J} = \mathcal{L} + I_d(B)$.

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**Theorem [Morey-U]**

Assume $\operatorname{codim} I = 2$ and $R/I$ is Cohen-Macaulay. If $I$ is $G_d$ and linearly presented, then $\mathcal{J}$ has the expected form:

$$\mathcal{R}(I) \cong S/\mathcal{L} + I_d(B) \quad \text{and} \quad A(X) \cong T/I_d(B)\,.$$

Moreover, both rings are Cohen-Macaulay.

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**Theorem [Buchsbaum-Eisenbud]**

$n$ is odd, $\varphi$ can be chosen to be alternating, and
$f_i = c\,(-1)^i\, \mathrm{Pf}_i(\varphi)$ for some $c \in k$ and $\forall i$.

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In this case, $\mathcal{J}$ does not have the expected form in general. In fact, the alternating property of $\varphi$ is responsible for 'unexpected' elements in $\mathcal{J}$:

Recall: $\underline{y} \cdot \varphi = \underline{x} \cdot B$ and $\varphi$ alternating

$$\Longrightarrow \quad \underline{x} \cdot B \cdot \underline{y}^t = \underline{y} \cdot \varphi \cdot \underline{y}^t = 0$$

$\Longrightarrow$ $B \cdot \underline{y}^t = 0$ and $\underline{y} \cdot B^t = 0$

Write

$$\mathcal{B} = \begin{bmatrix} \varphi & B^t \\ -B & 0 \end{bmatrix}$$

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$\Longrightarrow$ $\forall i$ with $1 \leq i \leq n$ : $y_i F = x_d F_i \in I_d(B) \cdot S$

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Recall: $y_i F \in I_d(B) \cdot T[\underline{x}] \;\; \forall i$ and $I_d(B) \subset T$

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**Main Theorem [Kustin-Polini-U]**

Assume $\operatorname{codim} I = 3$ and $R/I$ is Gorenstein. If $I$ is $G_d$ and linearly presented, then

$\mathcal{R}(I) \cong S/(\mathcal{L} + I_d(B)S + c(F)S)$ and $A(X) \cong T/(I_d(B) + c(F))$.

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Notice that in the setting of the Main Theorem,

- $I$ is of fiber type
- If $d$ is odd, then $F = 0$ and hence $\mathcal{J}$ has the expected form.

## Corollary

- $X$ is Cohen-Macaulay.
- $$\operatorname{depth} A(X) = \begin{cases} 1 & \text{if } d \text{ is odd and } n > d+1 \\ 2 & \text{if } d \text{ is even and } n > d+1 \end{cases}$$
  and $A(X)$ is Cohen-Macaulay otherwise.
- If $d$ is odd, then $I(X)$ has a linear resolution.

### Corollary

- $X$ is Cohen-Macaulay.
- $$\operatorname{depth} A(X) = \begin{cases} 1 & \text{if } d \text{ is odd and } n > d+1 \\ 2 & \text{if } d \text{ is even and } n > d+1 \end{cases}$$

  and $A(X)$ is Cohen-Macaulay otherwise.
- If $d$ is odd, then $I(X)$ has a linear resolution.

### Theorem [Polini-U]

$\mathcal{R}(I)$ is Cohen-Macaulay $\iff n \leq d+1$.

# The steps in the proof of the Main Theorem

(I) $I$ is of fiber type.

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(IV) Express $\deg X$ in terms of the $j$-multiplicity of the ideal $I$.

$\Longrightarrow$ $A(X)$ and $T/I_d(B) + c(F)$ have the same multiplicity.

$\Longrightarrow$ The two rings are isomorphic.

# (I) Bounding generator degrees of $\mathcal{J}$

$R = k[x_1, \ldots, x_d] \supset I = (f_1, \ldots, f_n)$, $f_i$ forms of the same degree $\delta$

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Hence:

$I$ is of fiber type $\iff$ $H^0_{\mathfrak{m}}(\mathcal{S})$ generated in $(\underline{x})$-degree 0

$\iff$ $H^0_{\mathfrak{m}}(\mathrm{Sym}_i(I(\delta)))$ generated in degree 0 $\forall i$.

Let

$$C_\bullet : \quad \dots \quad \to C_1 \to C_0 \to 0$$

be a homogeneous complex of finite $R$-modules with $H_0(C_\bullet) =: M$ and assume that $\dim H_j(C_\bullet) \leq j \ \ \forall j > 0$.

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**Proposition**

If $\operatorname{depth} C_j \geq j+1$ for $0 \leq j \leq d-1$, then

$$H^0_{\mathfrak{m}}(M) \text{ is concentrated in degrees } \leq b(C_d) - d,$$

where $b(C_d)$ denotes the largest generator degree of $C_d$.

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## Theorem

If $\mathrm{depth}\, C_j \geq \min\{j+2, d\}$ for $0 \leq j \leq d-1$, then

$$H^0_{\mathfrak{m}}(M) \text{ is generated in degrees } \leq b(C_{d-1}) - d + 1.$$

# Back to Rees algebras

$R = k[x_1, \ldots, x_d] \supset I = (f_1, \ldots, f_n)$ with $f_i$ forms of degree $\delta$.

Consider a minimal homogeneous presentation

$$\bigoplus R(-\varepsilon_j) \xrightarrow{\varphi} R^n \xrightarrow{[f_1 \ldots f_n]} I(\delta) \longrightarrow 0$$

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**Corollary**

Assume $\operatorname{codim} I = 2$ and $R/I$ is Cohen-Macaulay. If $I$ is $G_d$, then $\mathcal{A} = H^0_{\mathfrak{m}}(\mathcal{S})$ is concentrated in $\underline{x}$-degrees $\leq \sum_{j=1}^{d} \varepsilon_j - d$ and is generated in $\underline{x}$-degrees $\leq \sum_{j=1}^{d-1} \varepsilon_j - d + 1$.

Now assume that $\operatorname{codim} I = 3$ and $R/I$ is Gorenstein. In this case complexes of free modules were constructed by Kustin-U that are approximate resolutions of $\operatorname{Sym}_i(I)$ if $\mu(I_p) \leq \operatorname{codim} p + 1$ whenever $\operatorname{codim} p \leq d - 2$.

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## Corollary

Assume $\operatorname{codim} I = 3$ and $R/I$ is Gorenstein. In this case $\varepsilon_1 = \varepsilon_2 = \ldots =: \varepsilon$. If $I$ is $G_d$, then $\mathcal{A} = H^0_{\mathfrak{m}}(\mathcal{S})$ is concentrated in $\underline{x}$-degrees

$$\leq \begin{cases} d(\varepsilon - 1) & \text{if } d \text{ is odd} \\ d(\varepsilon - 1) + \frac{n-d-1}{2}\,\varepsilon & \text{if } d \text{ is even} \end{cases}$$

and is generated in $\underline{x}$-degrees $\leq (d-1)(\varepsilon - 1)$.

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In particular, if $I$ is linearly presented, then $I$ is of fiber type. So it remains to prove that $I(X) = I_d(B) + c(F)$.

# (II) The codimension of $I_d(B)$

$R = k[x_1, \ldots, x_d] \supset I$ a linearly presented ideal:

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**Theorem**

Assume $I_p$ is of linear type $\forall p \neq \mathfrak{m}$. If $\mathrm{Sym}_t(I(\delta))$, for some $t \gg 0$, has an approximate free resolution that is linear for the first $d$ steps, then

$$\mathcal{J} = \sqrt{\mathcal{L} + I_d(B)} \quad \text{and} \quad I(X) = \sqrt{I_d(B)}.$$

In particular, $\mathrm{codim}\, I_d(B) = \mathrm{codim}\, I(X)$.

The assumptions of the Theorem are satisfied if $\operatorname{codim} I = 3$, $R/I$ is Gorenstein, and $I$ is $G_d$ and linearly presented – as approximate resolution one takes the complexes of [Kustin-U]. Hence $\operatorname{codim} I_d(B) = \operatorname{codim} I(X)$.

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**Proof of the Theorem:** Recall

$$\mathcal{L} + I_d(B) \subset \mathcal{L} : \mathfrak{m} \subset \mathcal{J}$$

We prove that both containments are equalities up to radical.

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Substituting this into the first equation above gives

$$\mathcal{J} = \sqrt{\mathcal{L} + I_d(B)S}$$

□

# (III) The unmixedness of $I_d(B)+c(F)$, or minors of matrices annihilated by a column of indeterminates

Let $d$ and $n$ be any integers with $1 < d < n$, $T = k[y_1, \ldots, y_n]$, $B$ a $d \times n$ matrix of linear forms

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We construct complexes of free $T$-modules that are resolutions of $T/I_d(B)$ and $T/I_d(B) + c(F)$ if $\operatorname{codim} I_d(B) = n - d$. These resolutions show:

## Theorem

Assume $\operatorname{codim} I_d(B) = n - d$

- $\operatorname{pd}_T T/I_d(B) = \begin{cases} n-1 & \text{if } n-d \text{ is even} \\ n & \text{if } n-d \text{ is odd} \end{cases}$
- $\operatorname{pd}_T T/(I_d(B) + c(F)) = \begin{cases} n-1 & \text{if } n-d \text{ is even} \\ n-2 & \text{if } n-d \geq 3 \text{ is odd} \\ 1 & \text{if } n-d = 1 \end{cases}$
- $e(T/I_d(B)) = e(T/I_d(B) + c(F)) = \sum_{i=0}^{\lfloor \frac{n-d}{2} \rfloor} \binom{n-2-2i}{d-2}$

## Proposition

If $\operatorname{codim} I_d(B) = n - d$, then $I_d(B)_{y_i} = (I_d(B) + c(F))_{y_i}$ define Cohen-Macaulay rings $\forall i$.

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To show Cohen-Macaulayness, let $B_i$ be the matrix obtained from $B$ by deleting column $i$.

$$B \cdot \underline{y}^t = 0 \implies I_d(B)_{y_i} = I_d(B_i)_{y_i}$$

But $I_d(B_i)_{y_i}$ is the ideal of $d \times d$ minors of a $d \times n - 1$ matrix having codimension $\geq n - d = (n-1) - d + 1$. $\square$

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## Corollary

If $\operatorname{codim} I_d(B) = n - d$, then

$$\begin{aligned} I_d(B) + c(F) &= I_d(B)^{\mathrm{unm}} \\ &= I_d(B) \quad \text{iff } n - d \text{ is even.} \end{aligned}$$

# Back to Rees algebras

By now we have seen that

$$I_d(B) + c(F) \subset I(X)$$

are unmixed ideals of the same codimension. Therefore these ideals are equal if the rings they define have the same multiplicity. It remains to compute the multiplicity of $A(X) = T/I(X)$.

# (IV) The multiplicity of $A(X)$

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$$\Phi : \mathbb{P}_k^{d-1} \overset{[f_1:\ldots:f_n]}{- - \longrightarrow} \mathbb{P}_k^{n-1}$$

Assume: $\dim X = d - 1$

Write: $r := \deg \Phi = [k(\mathfrak{m}_\delta) : k(I_\delta)]$

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## Proposition

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- If $I$ is linearly presented, then $r = 1$ [Eisenbud-U, Simis]
- If $\operatorname{codim} I = 3$, $R/I$ is Gorenstein, and $I$ is $G_d$, then $R/\mathfrak{a} : I^{\infty} = R/\mathfrak{a} : I$ and free $R$-resolutions of the latter have been worked out [Kustin-U].

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Under the two combined assumptions, one obtains a formula for $e(A(X))$, which shows that $A(X)$ and $T/I_d(B) + c(F)$ have indeed the same multiplicity.