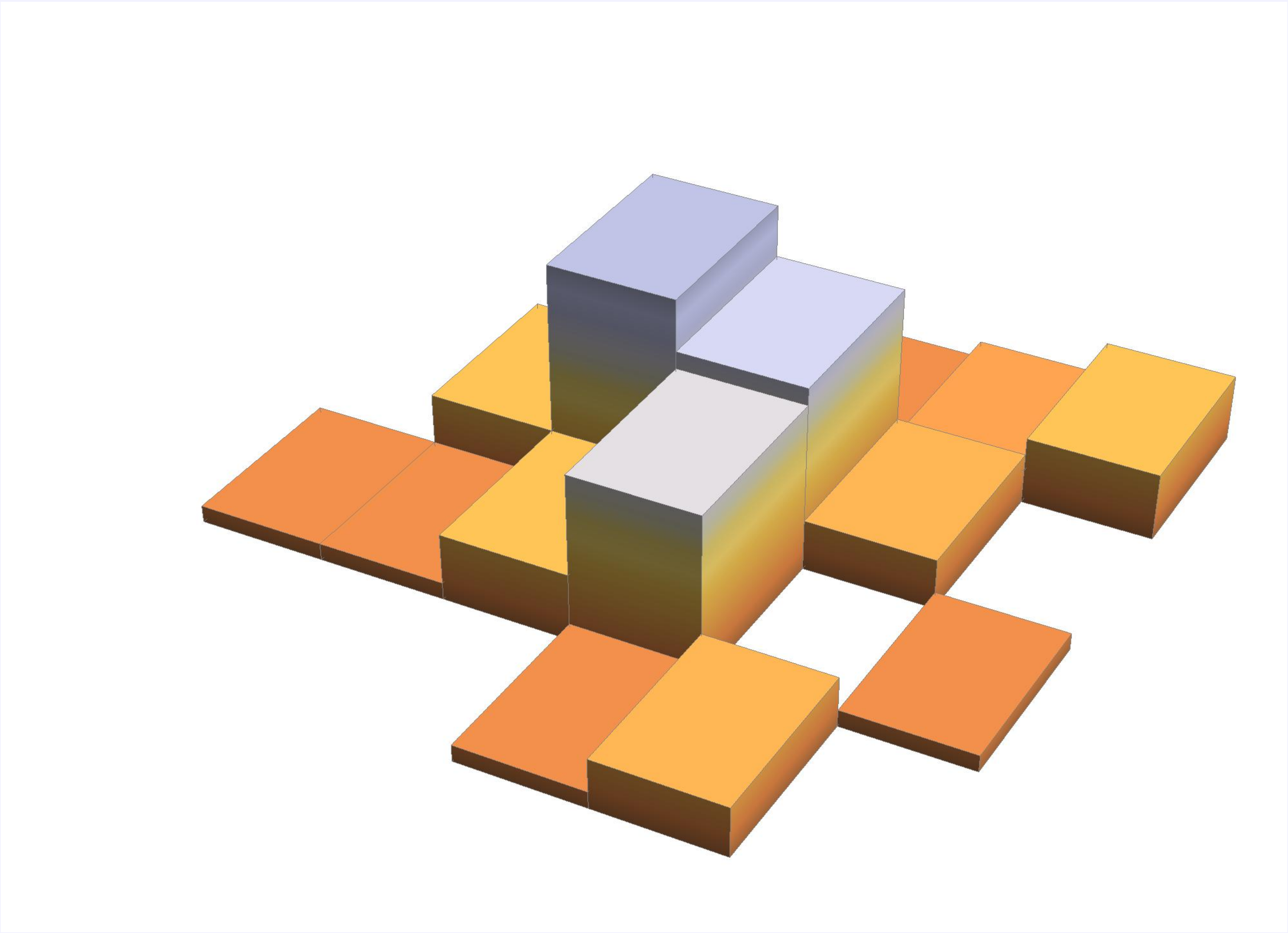
Noetherianity up to symmetry

Jan Draisma
TU Eindhoven and VU Amsterdam

Singular Landscapes
in honour of Bernard Teissier
Aussois, June 2015

A landscape, and a disclaimer



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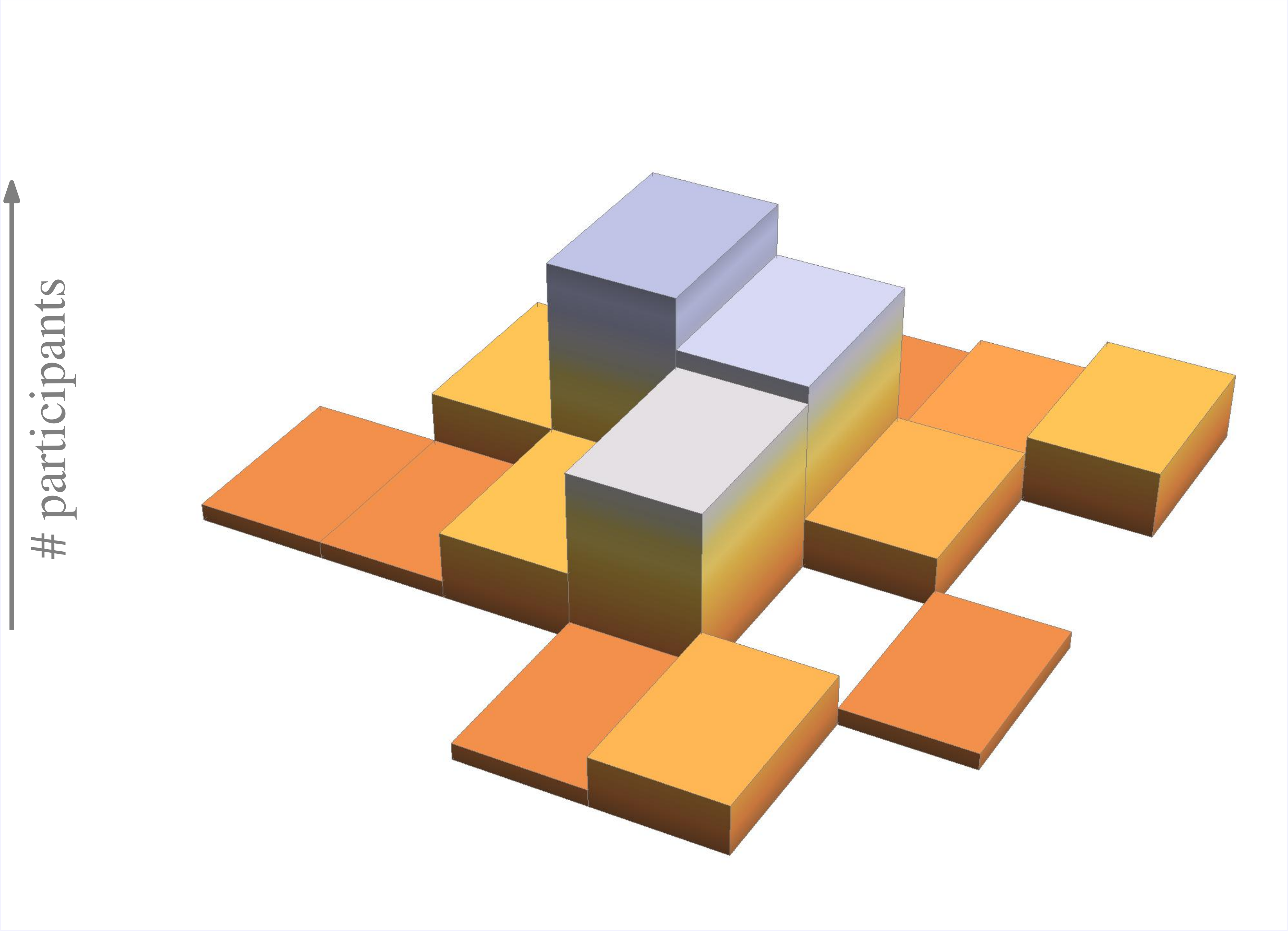

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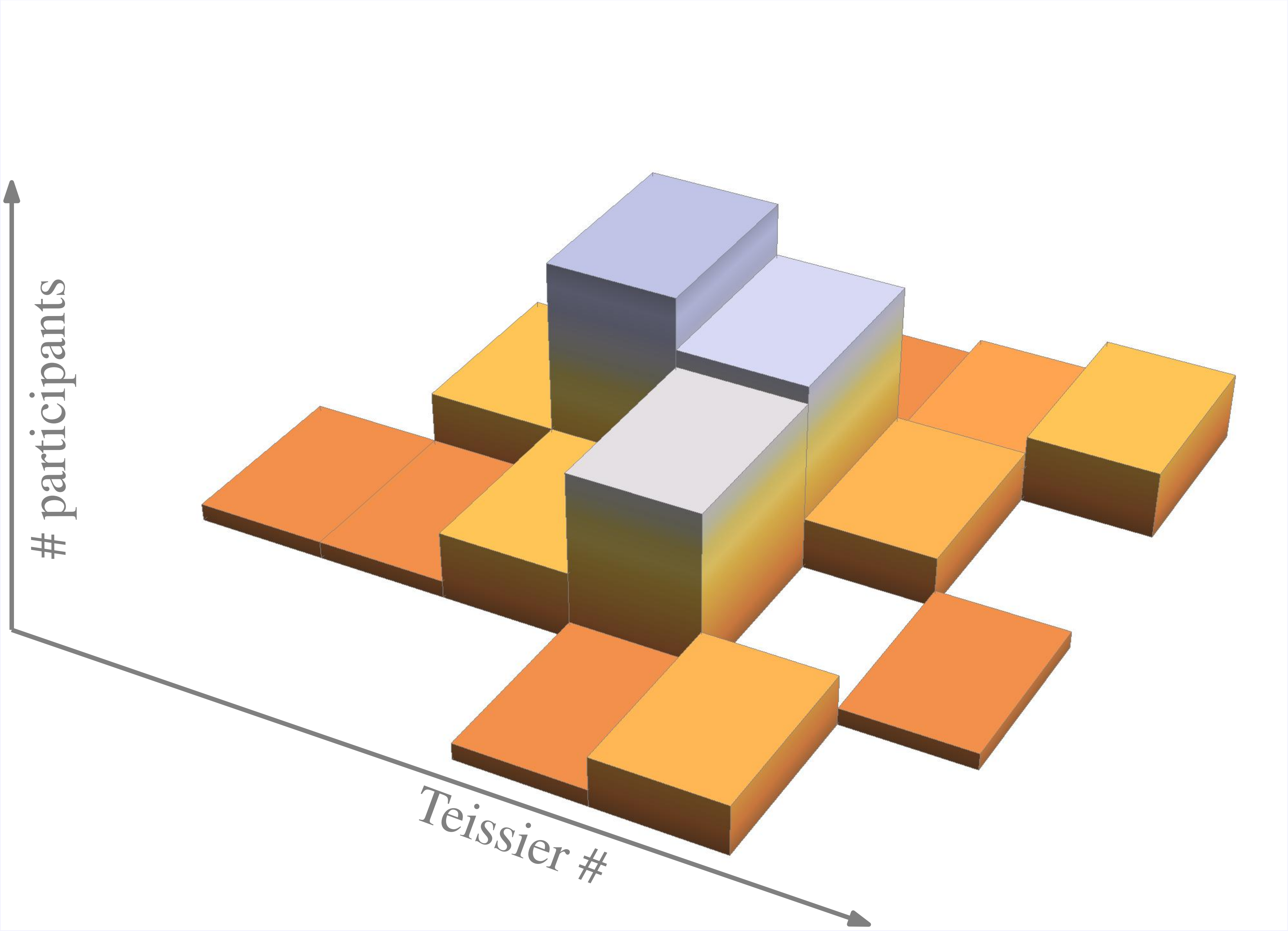

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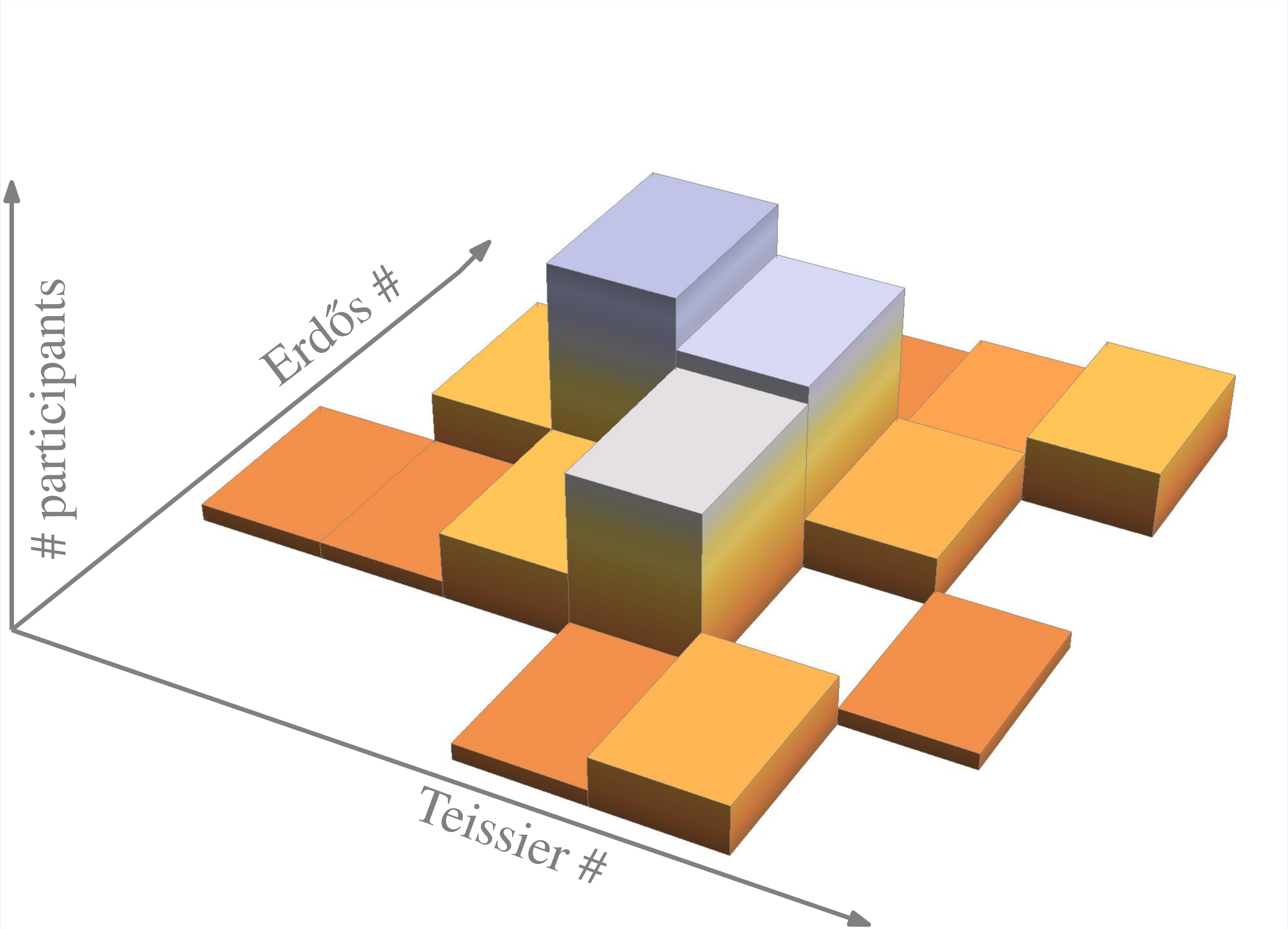

I. Equivariant Noetherianity

$K[x_1, x_2, x_3, \ldots]$

For K a field, $R := K[x_1, x_2, \ldots]$ is not a Noetherian ring ... but:

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Let $\mathrm{Sym}(\mathbb{N})$ act on R by $\pi x_i = x_{\pi(i)}$. Every chain $I_1 \subseteq I_2 \subseteq \ldots$ of $\mathrm{Sym}(\mathbb{N})$-stable ideals of R stabilises, i.e., I_n is constant for $n \gg 0$.

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Definition
Given a commutative ring R, a monoid Π, and an action of Π on R by algebra homomorphisms, R is Π-*Noetherian* if every chain $I_1 \subseteq I_2 \subseteq \ldots$ of Π-stable ideals stabilises.

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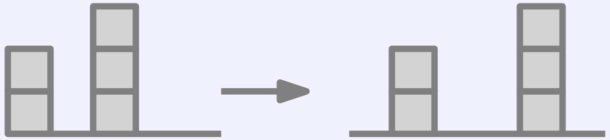
Equivalently:
- each Π-stable ideal I is generated by finitely many Π-orbits in R.
- R is a Noetherian $R * \Pi$-module (multiplication: $\pi * r = \pi(r) * \pi$).

Increasing maps

$\mathrm{Inc}(\mathbb{N}) := \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(1) < \pi(2) < \dots\}$ is a monoid, and it acts on $R = K[x_1, x_2, \dots]$ by $\pi x_i := x_{\pi(i)}$. For example, if $\pi : 1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 5, \dots$, then $\pi x_1^2 x_3^3 = x_2^2 x_5^2$.



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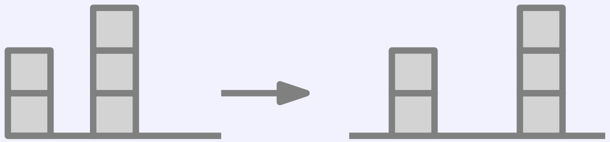

Cohen's theorem follows from:

Claim: $K[x_1, x_2, \ldots]$ is $\mathrm{Inc}(\mathbb{N})$-Noetherian.

Proof of Cohen's theorem

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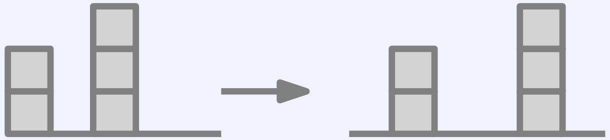

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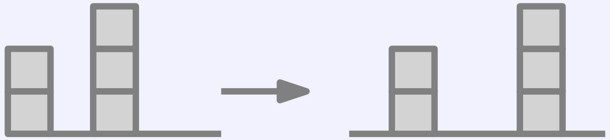

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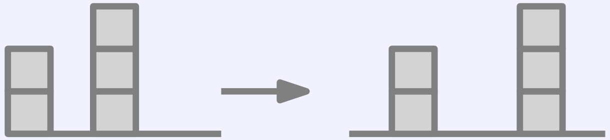

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Proof

- reduce to *monomial* ideals ($\mathrm{Inc}(\mathbb{N})$ preserves monomial orders).
- show that for any sequence $m_1, m_2, \ldots$ of monomials in x, there are $i < j, \pi \in \mathrm{Inc}(\mathbb{N}) : (\pi m_i) | m_j$ (*well-partial order*). □

Further examples: matrices

Theorem [Cohen 98/Hillar-Sullivant 09]

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If $\mathrm{char} K = 0$, then $K[x_{ij} \mid i, j \in \mathbb{N}]/((k+1) \times (k+1)$-minors of $x)$ *is* Inc($\mathbb{N}$)-Noetherian. (It is an invariant ring of GL_{k-1}.)

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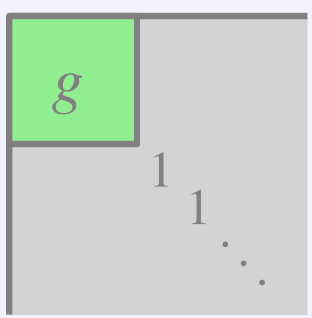
Theorem [Sam-Snowden 15]

If char$K = 0$, then $K[x_{ij} \mid i, j \in \mathbb{N}]$ is $\mathrm{GL}_{\mathbb{N}} \times \mathrm{GL}_{\mathbb{N}}$-Noetherian.

Here $\mathrm{GL}_{\mathbb{N}} = \{$ 

$\}$ acts by left and right multiplication.

Definition

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Lemma
• Π-equivariant images and finite unions of Π-Noetherian spaces are Π-Noetherian.
• If a group G acts on X by homeo, and $Z \subseteq X$ is H-Noetherian for a subgroup $H \subseteq G$, then $GZ := \bigcup_{g \in G} gZ$ is G-Noetherian.

Tuples of matrices

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Key notion

The *rank* of a tuple $(A_1, \ldots, A_p)$ is $\min\{\mathrm{rk} \sum_i c_i A_i \mid c \in \mathbb{P}^{p-1}\}$.

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Dichotomy
For $X \subseteq (K^{\mathbb{N}\times\mathbb{N}})^p$ closed and $\mathrm{GL}_{\mathbb{N}} \times \mathrm{GL}_{\mathbb{N}}$-stable, either:
1. $\sup_{A\in X} \mathrm{rk} A < \infty \rightsquigarrow$ can do induction on p; or
2. $\sup_{A\in X} \mathrm{rk} A = \infty \rightsquigarrow X = (K^{\mathbb{N}\times\mathbb{N}})^p$.

II. Why?

Motivating question

$X_1, X_2, \ldots$ algebraic varieties

$X_n \subseteq A_n$ closed embedding $\rightsquigarrow$ *stabilise* for $n \gg 0$?

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Running example

$A_n = K^{n \times n}$ ($n \times n$-matrices over a field K)

$X_n = \{x \in A_n \mid \operatorname{rank} x \leq 1\}$

defined by equations $x_{ij}x_{kl} - x_{il}x_{kj} = 0$ for all $n \geq 2$

Passing to a limit

Set-up

A_n a finite-dimensional vector space, $\pi_n : A_{n+1} \to A_n$ linear

$X_n \subseteq A_n$ a closed subvariety, fitting in a commutative diagram

$$\begin{array}{ccccccc} A_1 & \overset{\pi_1}{\twoheadleftarrow} & A_2 & \overset{\pi_2}{\twoheadleftarrow} & A_3 & \overset{\pi_3}{\twoheadleftarrow} & \dots \\ \bigcup & & \bigcup & & \bigcup & & \\ X_1 & \longleftarrow & X_2 & \longleftarrow & X_3 & \longleftarrow & \dots \end{array}$$

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${\color{red}A_\infty}$: dual of a countable-dimensional space

${\color{red}X_\infty}$: ∞-dim variety

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Running example

$A_n = K^{n \times n} \supseteq X_n = \{\text{rank} \leq 1 \text{ matrices}\}$

π_n forgets the last row and column

$A_\infty = {\color{red}K^{\mathbb{N} \times \mathbb{N}}}$ space with coordinates x_{ij}, $i, j \in \mathbb{N}$

$X_\infty = \{\mathbb{N} \times \mathbb{N} \text{ rank} \leq 1 \text{ matrices}\}$

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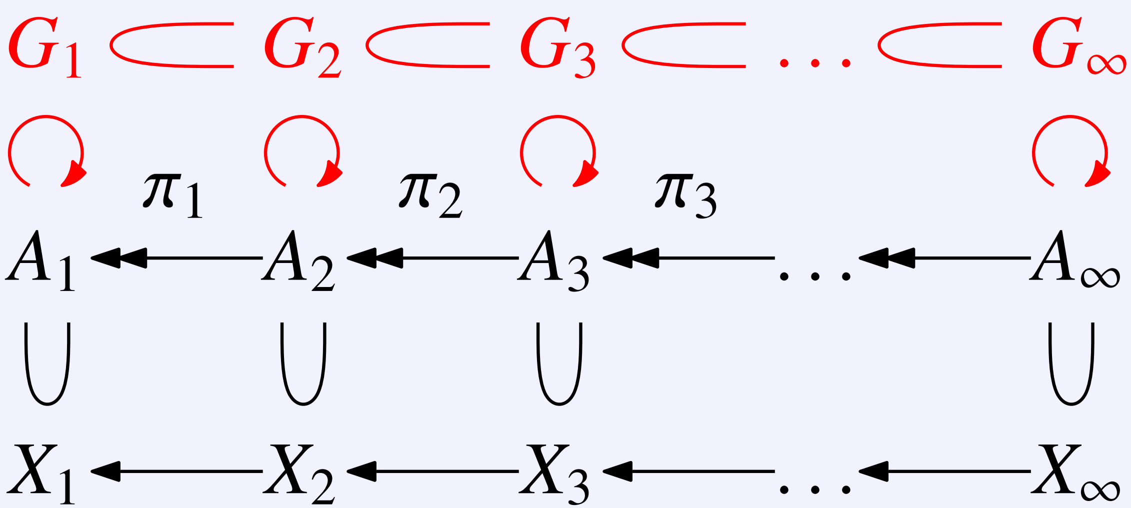

Assumptions:

- G_n acts linearly
- G_n preserves X_n
- π_n is G_n-equivariant

$\rightsquigarrow$ *G_∞ acts on A_∞ and preserves X_∞*

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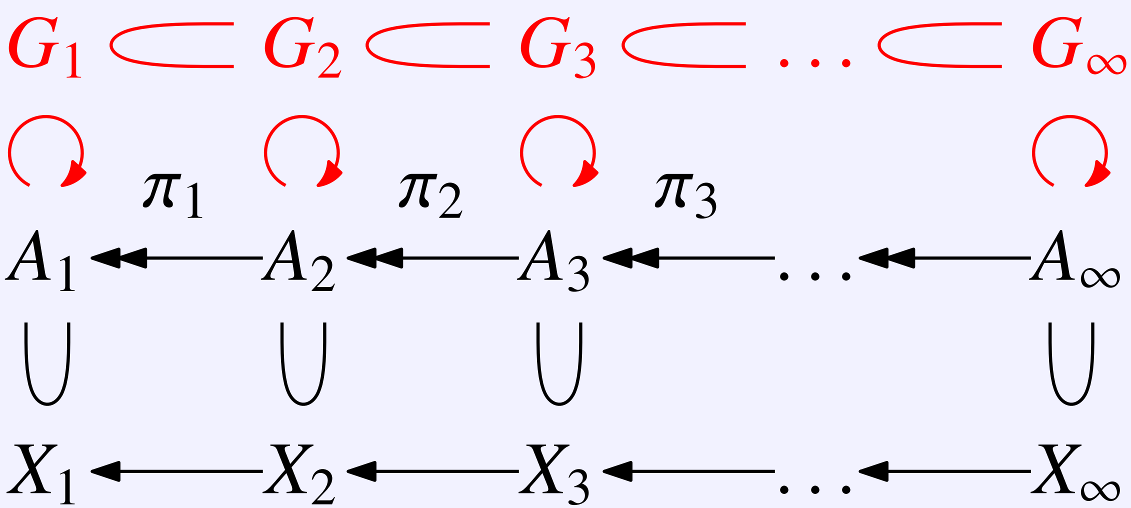

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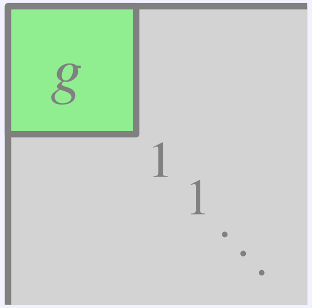
Running example

$A_n = K^{n \times n}$, $G_n = \mathrm{GL}_n(K)$ acting by $(g, a) \mapsto gag^{-1}$
$G_\infty = \mathrm{GL}_{\mathbb{N}}(K)$, preserves X_∞



Summary

X_∞ is a variety in the vector space A_∞ with *countably many coordinates*. If f is a polynomial that vanishes everywhere on X_∞, then so is $gf := f \circ g^{-1}$ for all $g \in G_\infty$.

Formalising stabilisation

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Question (with many variants)

Is X_∞ the common zero set of finitely many *orbits* $G_\infty f_1, \ldots, G_\infty f_s$ of polynomial equations? Typical proof strategy: find a G_∞-Noetherian subvariety Y_∞ of A_∞ containing X_∞.

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Example: rank-one matrices
X_∞ is defined by the $\mathrm{GL}_{\mathbb{N}}(K)$-orbit of $x_{11}x_{22} - x_{12}x_{21}$ so the family $\{X_n\}_n$ stabilises.

III. Topics

Definition

Rank of $\omega \in V_1 \otimes \cdots \otimes V_n$ is the minimal k in any expression $\omega = \sum_{i=1}^{k} v_{i1} \otimes \cdots \otimes v_{in}$. (*For* $n = 2$ *this is matrix rank.*)

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Theorem [D-Kuttler, 2014]

For any fixed k there is a d, *independent of n and the V_i*, such that $\overline{\{\omega \mid \text{rank}\, \omega \leq k\}}$ is defined by polynomials of degree $\leq d$.

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Table

k	0	1	2	3	4
d	1	2	$3^\dagger$	$4^\bullet$	$\geq 9^*$

† [Landsberg-Manivel, 2004]

• [Qi, 2014]

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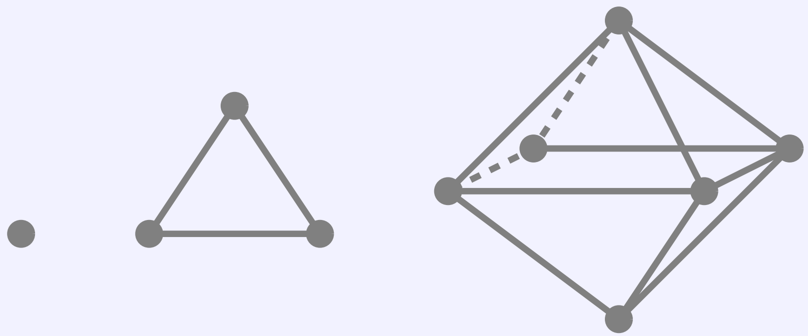
Proof set-up

$$A_n = (K^{k+1})^{\otimes n} \supseteq X_n = \overline{\{\text{rank} \leq k\}} \circlearrowleft G_n = S_n \ltimes \mathrm{GL}_{k+1}^n$$

$$\pi_n : A_{n+1} \to A_n, (v_1 \otimes \cdots \otimes v_{n+1}) \mapsto x_0(v_{n+1}) \cdot v_1 \otimes \cdots \otimes v_n$$

Second hypersimplex

$P_n := \{v_{ij} = e_i + e_j \mid 1 \leq i \neq j \leq n\}$



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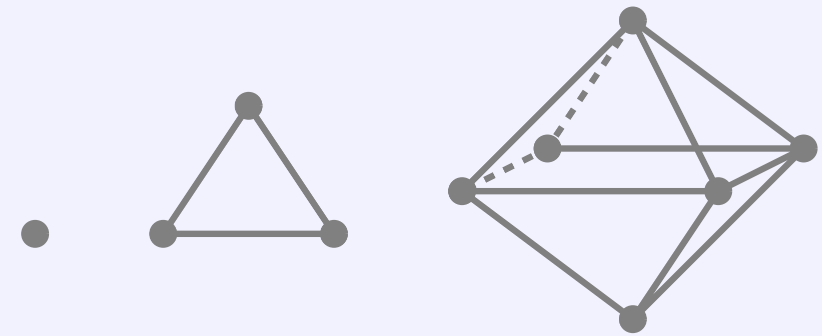

Theorem [De Loera-Sturmfels-Thomas 1995]

P_n has a Markov basis consisting of *moves* $v_{ij} + v_{kl} \to v_{il} + v_{kj}$ and $v_{ij} \to v_{ji}$ for i, j, k, l distinct; i.e., if $\sum_{ij} c_{ij} v_{ij} = \sum_{ij} d_{ij} v_{ij}$ with $c_{ij}, d_{ij} \in \mathbb{Z}_{\geq 0}$, then the expressions are connected by such moves without creating negative coefficients.

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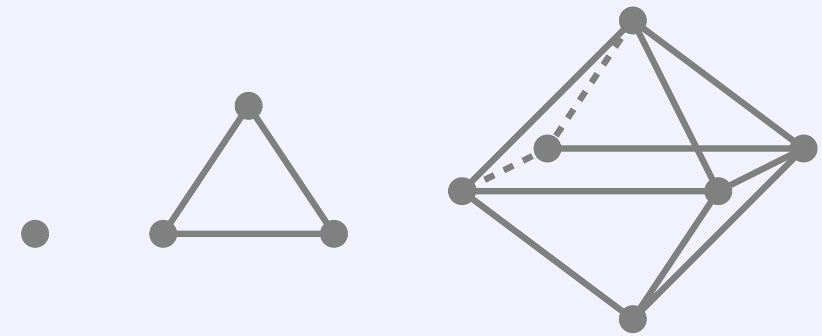

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Theorem [D-Eggermont-Krone-Leykin *2013*]

For *any* family ($P_n \subseteq \mathbb{Z}^{k \times n}$), if $P_n = S_n P_{n_0}$ for $n \geq n_0$, then $\exists n_1$: for $n \geq n_1$ has a Markov basis M_n with $M_n = S_n M_{n_1}$.

$\rightsquigarrow$ *we also have an algorithm for computing* n_1 *and* M_{n_1}

Grassmannians

$\mathbf{Gr}_k(V) \subseteq \mathbb{P}(\bigwedge^k V)$ is *functorial in V*, and the "Hodge dual" $\bigwedge^k V \to \bigwedge^{n-k} V^*$ with $\dim V = n$ maps $\mathbf{Gr}_k(V) \to \mathbf{Gr}_{n-k}(V^*)$.

Grassmannians

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A sequence $(\mathbf{X}_k)_k$ of rules $\mathbf{X}_k : V \mapsto X_k(V) \subseteq \mathbb{P}(\bigwedge^k(V))$ with these two properties is a *Plücker variety*.

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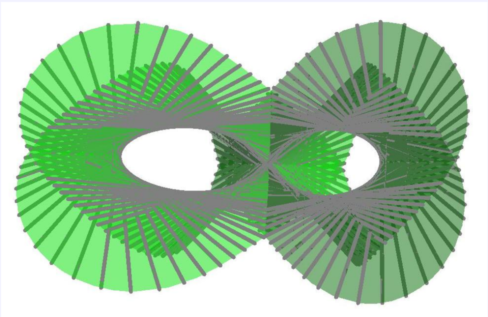

Construction of new Plücker varieties

tangential variety, secant variety, etc.

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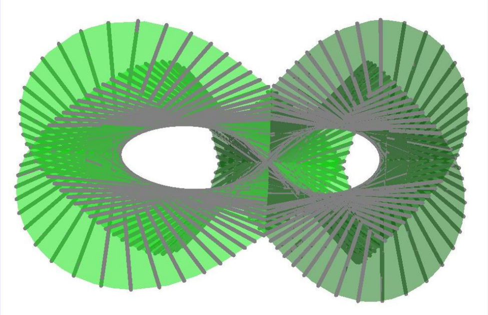

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Theorem [D-Eggermont 2014]

For a *bounded* Plücker variety $\mathbf{X}$, $(\mathbf{X}_k(K^n))_{k,n-k}$ is defined in bounded degree.

The infinite wedge

$$V_\infty := \langle \ldots, x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3, \ldots \rangle_K$$
$$V_{n,p} := \langle x_{-n}, \ldots, x_{-1}, x_1, \ldots, x_p \rangle \subseteq V_\infty$$

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$$\begin{array}{cc} \bigwedge^p V_{np} & \bigwedge^{p+1} V_{n,p+1} \\ \downarrow & \\ \bigwedge^p V_{n+1,p} & \end{array}$$

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$\bigwedge^{\infty/2} V_\infty := \lim_{\rightarrow} \bigwedge^p V_{n,p}$ *the infinite wedge* (charge-0 part);
basis $\{x_I := x_{i_1} \wedge x_{i_2} \wedge \cdots\}_I$, $I = \{i_1 < i_2 < \dots\}$, $i_k = k$ for $k \gg 0$

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On $\bigwedge^{\infty/2} V_\infty$ *acts* $\mathrm{GL}_\infty := \bigcup_{n,p} \mathrm{GL}(V_{n,p})$.

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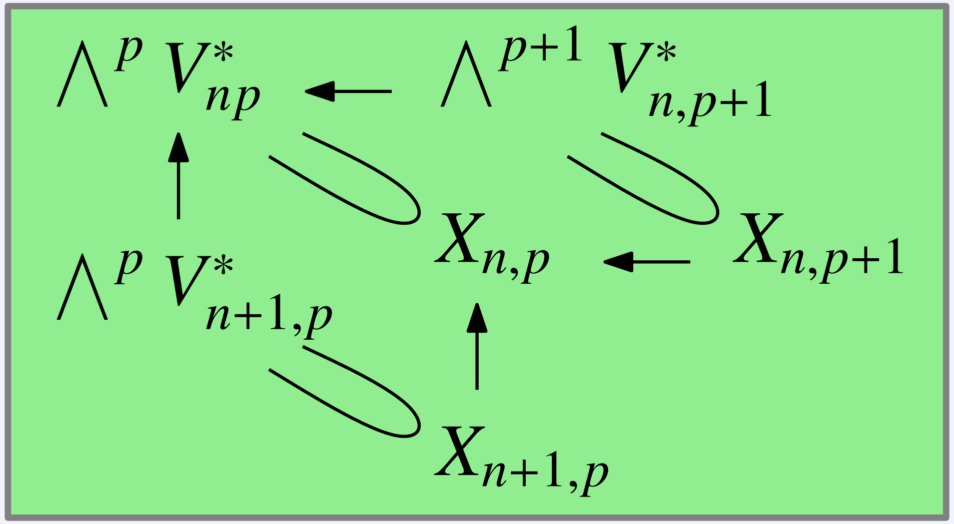

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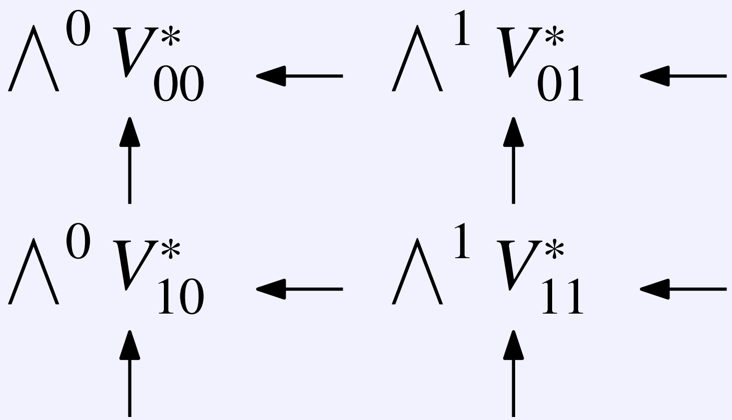

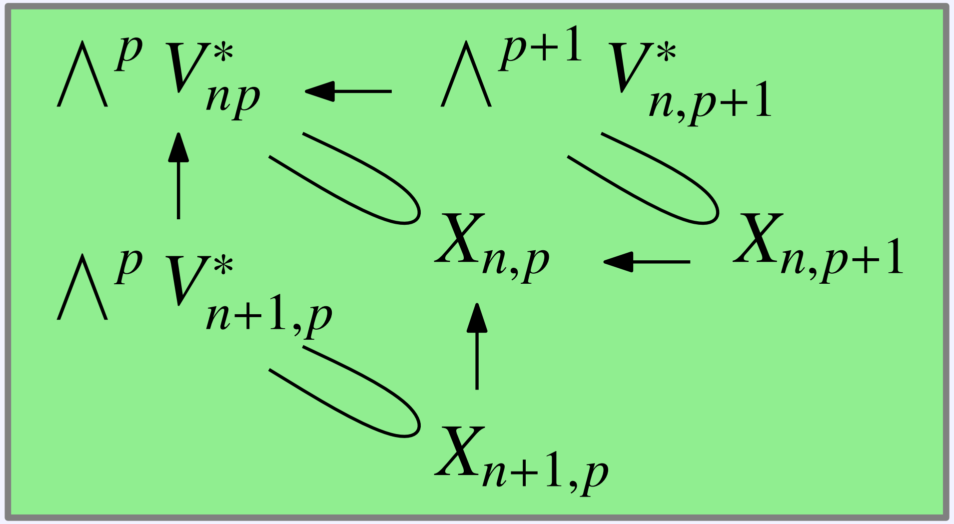

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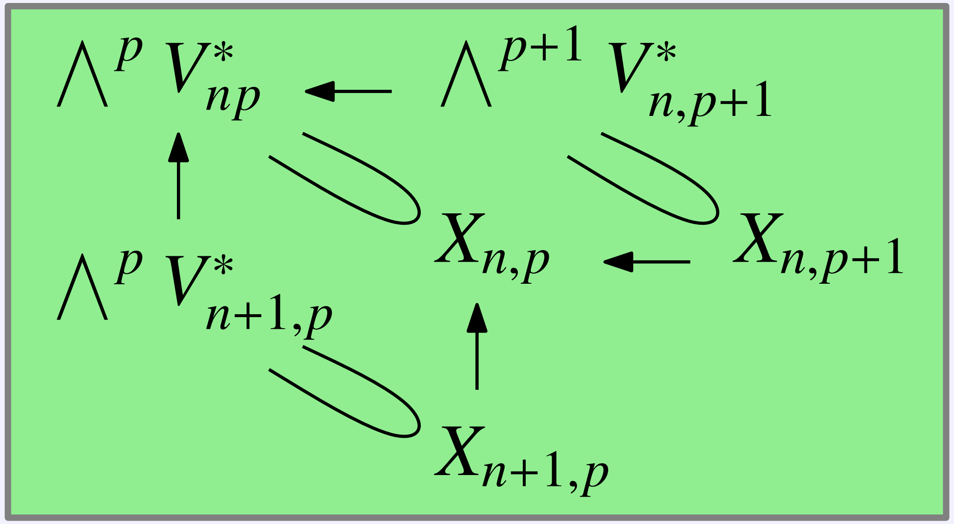

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Theorem
$\mathbf{X}$ *bounded* $\Rightarrow$ $\mathbf{X}_\infty$ cut out by finitely many GL_∞-orbits of eqs.

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- By boundedness, $\mathbf{X}_\infty \subseteq \mathbf{Y}_\infty^{(k)}$, where latter is defined in the dual infinite wedge by the orbit of a certain $(2k \times 2k)$-Pfaffian.

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- So Z is $\mathrm{GL}_\mathbb{N} \times \mathrm{GL}_\mathbb{N}$-Noetharian, and $\mathrm{GL}_\infty Z$ is GL_∞-Noetherian, and so is $\mathbf{Y}_\infty^{(k)}$, and hence $\mathbf{X}_\infty$ is defined by finitely many further GL_∞-orbits of equations. $\square$

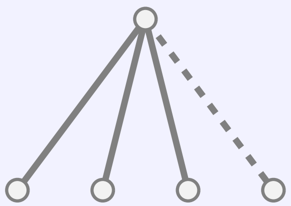
IV. Further areas

Stabilisation in other areas

Algebraic statistics

families of graphical models where the graph grows

[Hillar-Sullivant, Takemura, Yoshida, D-Eggermont,. . .]



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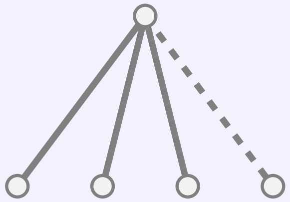

Commutative algebra and representation theory

higher syzygies, sequences of modules

[Sam-Snowden, Church-Ellenberg-Farb, . . .]

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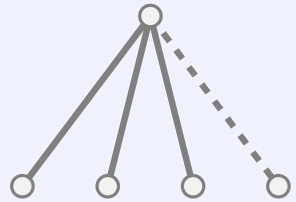

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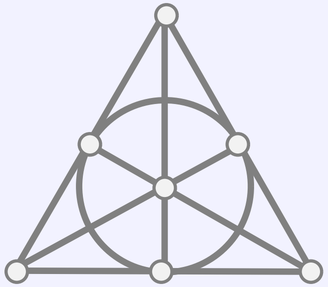
[Sam-Snowden, Church-Ellenberg-Farb, . . .]

Combinatorics

matroid minor theory

[Geelen-Gerards-Whittle, . . .]



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Keywords include FI-modules (Church-Ellenberg-Farb), Delta-modules (Snowden), twisted commutative algebras (Sam-Snowden), equivariant Noetherianity, and equivariant Gröbner bases.

Thank you!