Resolution by alterations

M. Temkin

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Conference in honor of Bernard Tessier

Plan

1. Introduction
   - Outline
   - l'-altered resolution theorem
2. de Jong's Galois alteration theorem
3. Gabber's torification theorem
4. Proofs of l'-alteration results
5. Towards $\mathrm{char}(X)$-alteration

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- Let $T \subseteq P = \{2, 3, 5, \dots\}$ be a set of primes. Then f is a *T-alteration* if any prime dividing $\deg(f)$ is in T. Examples:
  - If $T = \emptyset$ then f is a *modification*.
  - If $T = \{p\}$ then f is a *p-alteration*.
  - If $T = P \setminus \{l\}$ then f is an *l'-alteration*.

Main results

- We will discuss the following results from [IT14] (exposé X in "Travaux de Gabber sur ...", Asterisque 363-364):
  (0) l'-altered resolution of varieties (Illusie-T, after Gabber).
  (1) l'-altered semistable reduction over an excellent curve (Illusie-T, after Gabber).
  (d) l'-altered resolution of a morphism $X \to S$ (Illusie-T).

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  - (d) l'-altered resolution of a morphism $X \to S$ (Illusie-T).
- (d) subsumes (0) and (1) when $d = \dim(S) \leq 1$. The main ingredients in all three results are de Jong's Galois alteration theorem and Gabber's torification theorem, but the scheme used to prove (d) differs from that for (0), (1).

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- (d) subsumes (0) and (1) when $d = \dim(S) \leq 1$. The main ingredients in all three results are de Jong's Galois alteration theorem and Gabber's torification theorem, but the scheme used to prove (d) differs from that for (0), (1).
- In the end, we will discuss a new approach (work in progress) that aims to upgrade (d), and hence also (0) and (1), to $\mathrm{char}(X)$-alterations (e.g. p-alteration when $X \to \mathrm{Spec}(\mathbf{Z})$ factors through $\mathrm{Spec}(\mathbf{F}_p)$).

Resolution of morphism

- The aim is to approximate a morphism $f\colon X \to S$ with a mildly singular $f' : X' \to S'$: find suitable coverings $\alpha\colon S' \to S$ and $\beta\colon X' \to X$, and a morphism f' compatible with f

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- Typically, the base change α is an alteration, at least if one wants f' to have reduced fibers.
- In the ideal situation, β' is a modification of an irreducible component of $X \times_S S'$.
- Can also consider a divisor in X (omitted for simplicity). If α, β' are T-alterations then f' is called a *T-alteration* of f.

Examples of relative resolution

Here are the main examples that were known when $\dim(S) \geq 1$.

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By an example of Karu, semistable alterations do not have to exist when $\dim(f) \geq 2$ and $\dim(S) \geq 2$. So, the log smoothness condition seems to be the one with mildest possible singularities.

l'-altered resolution of morphisms

Definition

An integral X is *universally T-resolvable* if for any alteration $X_1 \to X$ and a closed subset $Z_1 \subsetneq X_1$ there exists a T-alteration $f\colon X' \to X_1$ such that X' is regular and $f^{-1}(Z_1)$ is an snc divisor.

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Theorem

Assume that $f\colon X \to S$ is of finite type, l is a prime invertible on S and S is universally l'-resolvable. Then X is universally l'-resolvable and for any closed $Z \subsetneq X$ where exist regular schemes with snc divisors (X', Z') and (S', W') and l'-alteration $f'\colon X' \to S'$ of f such that $Z' = \beta^{-1}(Z) \cup f'^{-1}(W')$ and $(X', Z') \to (S', W')$ is log smooth.

Addenda

- The same proof shows that if $\mathrm{char} = 0$ then one can even achieve that both α and β are modifications.
- Using log geometry it is easy to see that enlarging α (and loosing the l'-property and regularity of X') one can make f' saturated (flat with reduced fibers). In $\mathrm{char} = 0$ this extends Abramovich-Karu to the case when S is not a field.
- If $S = \mathrm{Spec}(k)$ for a perfect field k then can also achieve that β is separable.

A low-dimensional S

(0) If $\dim(S) = 0$ then $S = \mathrm{Spec}(k)$ for a field k, $W = \emptyset$ and $(X', Z') \to S'$ is log smooth iff X' is smooth and Z' is snc, so we obtain l'-altered resolution of varieties with closed subsets. (In addition, X' is smooth over a finite extension $S' = \mathrm{Spec}(k')$, but this is not a real strengthening.)

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(1) If $\dim(S) = 1$ then S' is a regular curve and a saturated log smooth $(X', Z') \to (S', W')$ is necessarily semistable (also, saturation is very simple here). So, we obtain altered semistable modification of $X \to S$ with a closed $Z \hookrightarrow X$, plus precise control on the toroidal divisor Z': the horizontal component is the preimage of Z. (In the l'-altered version there might be non-reduced fibers.)

Formulations

Theorem

Let X be a variety over a field k and $Z \subsetneq X$ a closed subset. Then there exists a Galois alteration $f\colon X' \to X$ such that X' is regular and $Z' = f^{-1}(Z)$ is an snc divisor. If k is perfect, one can also achieve that f is generically étale (or separable).

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- de Jong also proves a similar theorem for X of finite type over S assuming that any Galois alteration of S can be resolved by a larger Galois alteration of S.
- If S is a trait, a semistable reduction analogue of this theorem was proved by Gabber-Vidal in 2003.

The induction step

de Jong's theorem is proved by induction on dimension. One finds a curve fibration $g\colon X \to Y$, applies induction assumption to Y and resolves the relative curve g by the following theorem:

Theorem

If $g\colon X \to Y$ is a proper morphism of integral qe schemes and the generic fiber is a curve then there exists a semistable alteration $g'\colon X' \to Y'$ such that α is a Galois alteration and β' is a modification.

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- The proof uses moduli space of proper curves and a three points stabilization trick.
- de Jong also treats divisors and group actions (critical, but skipped for shortness).
- The main inconvenience is that f should be proper; sometimes one has to work hard for compactification.

Stable modification theorem

- A slightly stronger stable modification theorem was proved in [Tem10]. It applies to any relative curve f (even non-separated) and claims that there exists a unique minimal semistable (or stable) modification β' once a sufficiently large α is chosen.
- The proof is by a completely different technique. First one proves this over $S_v = \mathrm{Spec}(R_v)$ for a valuation ring R_v. Since the Riemann-Zariski space of S is quasi-compact this implies that stable modification exists after a base change α' which is a Zariski-alteration covering of S. Since the stable modification is unique, this descends to an actual alteration α.

Toroidal schemes and log smooth morphisms

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- Assume X is regular. Then D is snc iff (X, D) is toroidal.
- If (X, D) is log regular, then X possesses a canonical "combinatorial" resolution $f_{(X,D)} : X' \to X$ (Kato, Niziol, Gabber).

Log smooth morphisms

- Informally speaking, *log smooth morphisms* are the toroidal ones (i.e. given by monomials formally-locally) with a restriction on p-th powers for non-invertible primes p (e.g. Frobenius is not log smooth).
- Log smoothness is a more flexible and functorial notion than toroidality (e.g. it makes sense for not log regular log schemes).

Toroidal action

Assume that a finite group G acts on a toroidal scheme (X, D).

- The action is *tame* at $x \in X$ if the stabilizer G_x has order invertible in $k(x)$.
- The action is *simple* at x if G_x preserves the components of D at x (i.e. acts trivially on the monoid $\overline{M}_x$).
- The action is *toroidal* (very tame in [IT14]) at x if it is simple and tame at x and G_x acts trivially on the log stratum of D at x.

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Theorem

If G acts toroidally on (X, D) then $(X/G, D/G)$ is toroidal and $(X, D) \to (X/G, D/G)$ is log smooth.

Torification theorem

Definition

A *torification* of an action of G on a toroidal scheme (X, D) is a G-equivariant toroidal morphism $(X', D') \to (X, D)$ such that $X' \to X$ is a modification and the action on (X', D') is toroidal.

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- Locally one can simply increase D (i.e. $X' = X$), but this is not canonical and does not globalize.
- Nevertheless, in two existing proofs this is a start point, and after blowings up in non-reduced centers one manages to remove the added components.
- No proof going by blowing up smooth centers is known!

Gabber's proof

Gabber's proof is in [IT14] and it is very complicated. Main idea: find a resolution $Y' \to Y = X/G$ and pull it back to X. Since Y is only locally (and non-canonically) toroidal, one can resolve Y locally, but it is difficult to show that this is canonical and hence globalize to the whole Y. Gabber does this by lifting the situation to characteristic zero and using that there exists canonical resolution there.

"Its torific" (from an email of Abramovich to de Jong)

Another, and historically first, approach is by blowing up an explicit so-called *torific* ideal associated to (X, D, G).

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- Abramovich-Karu-Matsuki-Włodarczyk (2001) established torification for an action of G_m on toroidal varieties of $\mathrm{char} = 0$. This is an essential step in establishing factorization of birational maps between regular varieties.

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- Abramovich-T (2015) used torific blow up to prove torification in general. Here G is any diagonalizable group and (I think) the argument is simpler than Gabber's. (We use this with $G = G_m$ to extend factorization to general toroidal schemes, assuming desingularization.)

First method: single factorization

Gabber's proof of l'-resolution when $\dim(S) \leq 1$ goes as follows:

(0) Preparation: can assume $X \to S$ is proper, etc.

(1) By de Jong's (resp. Gabber-Vidal) theorem find a G-Galois alteration $f' : X' \to S'$ of f which is smooth (resp. semistable).

(2) Torify the action of an l-Sylow $G_l \subseteq G$ on (X', Z') by an additional modification.

(3) Divide (X', Z') by G_l and resolve the singularities of the quotient toroidal scheme.

Second method: step-wise factorization

The proof of Illusie-T applies to any universally l'-resolvable S and runs as follows:

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(1) Establish the case of $\dim(f) = 1$: apply de Jong's semistable modification of curves and divide by an l-Sylow subgroup using the torification theorem.

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Gabber's method divides by l-Sylow once, while we divide on each step. In particular, the resulting log smooth map $(X', Z') \to (S', W')$ is a composition of log smooth but not semistable curves (their fibers may be non-reduced). This is one of advantages of working with log smooth maps.

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- I am going to talk about a few new ideas that look very promising, but nothing has been written down so far!
- It seems certain that one can achieve p-altered local uniformization in this way, but it is too early to talk about a global result.

Distillation of alterations

The main idea is to use the following

Conjecture (Tame distillation of alterations)

Any alteration $Y \to X$ can be enlarged to a Galois alteration $Y' \to X$ that splits to a tame alteration $Y' \to X'$ and a $\mathrm{char}(X)$-alteration $X' \to X$.

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- (It seems that) once the distillation theorem is proved, one can strengthen l'-altered resolution of morphisms to $\mathrm{char}(X)$-altered resolution assuming the base S is universally $\mathrm{char}(X)$-resolvable.
- The proof is the same, but thanks to distillation we divide at once by the whole non-$\mathrm{char}(X)$ packet.
- This time it is critical to divide on each step, because the control on the alteration is tight only for relative curves: β' is a modification and for the base we do have the induction assumption.
- This approach unifies the general and the $\mathrm{char} = 0$ cases.

Why is distillation plausible?

- By Pank's theorem any finite extension of henselian valued fields L/K can be enlarged to a finite extension L'/K that splits to a tame extension L'/K' and a purely wild extension K'/K (so, $[K' : K] = p^n$ where p is the residue characteristic).

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- For any valuation v of the Riemann-Zariski space of X we obtain a (non-Galois) p-extension $K_v/k(X)$ that provides tame distillation of $Y \times_X \mathrm{Spec}(R_v) \to \mathrm{Spec}(R_v)$. By quasi-compactness of $\mathrm{RZ}(X)$, only finitely many K_v are needed to distill any valuation of $\mathrm{RZ}(X)$.

Why is distillation plausible?

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- If $K_{v_1},\dots,.K_{V_n}$ lie in a $\mathrm{char}(X)$-extension K'/K then K' induces a distillation of X.

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- This both indicates that the distillation conjecture should hold and suggests a way to by-pass the obstacle: deform K_{v_i} slightly (e.g. in the decompletion stage) so that it still distills valuations near v_i and K_{v_i}'s become independent.

Happy Birthday Bernard!

- Luc Illusie and Michael Temkin, Gabber's modification theorem (log smooth case), exposé X, Astérisque **363-364** (2014), 169–216.
- Michael Temkin, Stable modification of relative curves, J. Algebraic Geom. **19** (2010), no. 4, 603–677. MR 2669727 (2011j:14064)