Sets with few rational points

Aussois
June 2015

Bernard Teissier's Birthday

Introduction: How to bound the number of rational points in some given set?
Since this question is in general too difficult to deal with, we only look at rational points of bounded height.

Definition: $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$ is of height $H(x)$ bounded by T when $\forall i \in \{1, \dots, n\}$, $x_i = a_i/b_i$, $(a_i, b_i) \in \mathbb{Z} \times \mathbb{Z}^*$, $a_i \wedge b_i = 1$ and $|a_i| \leq T$, $|b_i| \leq T$.

The first general result to mention in this direction is the Pila & Wilkie theorem, which needs some preliminary notation.

Let $\mathcal{S}$ be an o-minimal structure expanding $(\mathbb{R}, \times, +, -, 0, 1)$.
We denote, for $X \in \mathcal{S}$, $X^{alg} = \{x \in X;\ \exists\ \gamma$ a pure one dimensional semialgebraic subset of $\mathbb{R}^n$ / $x \in \gamma \subseteq X\}$.
We denote $X^{trans} := X \setminus X^{alg}$ and $X^{trans}(\mathbb{Q}, T) = \{x \in X^{trans} \cap \mathbb{Q}^n \,/\, H(x) \leq T\}$.
We look at rational points in X out of semi-algebraic pieces, that usually contain too many such points.

Theorem (Pila & Wilkie, Duke 2006): $\forall \varepsilon > 0\ \exists\ C_{X,\varepsilon} > 0$, $\# X^{trans}(\mathbb{Q}, T) \leq C_{X,\varepsilon} \cdot T^{\varepsilon}$

Remark. This result is best possible in the following sense: there exists, for any $\varepsilon :]1, +\infty[\to \mathbb{R}$, $\varepsilon \to 0$, $f : [0,1] \to \mathbb{R}$ analytic and transcendent, there exists a sequence $T_j \to +\infty$ /
$\# \Gamma_f(\mathbb{Q}, T) \geq T_j^{\varepsilon(T_j)}$, where Γ_f is the graph of f.
(cf Pila 2004, Surroca 2002)

Wilkie conjecture: $\mathbb{R}exp$ the expansion of $(\mathbb{R}, 0, 1, +, -, \cdot)$ by $\Gamma_{\exp : \mathbb{R} \to \mathbb{R}}$, then $\forall X$ definable in $\mathbb{R}exp$, $X^{trans}(\mathbb{Q}, T) \leq C_X \log^{\beta}(T)$, for some β.

What is known so far: the conjecture is true for pfaffian curves, some graphs of analytic transcendent functions on a compact set.

Question: to what extend compactness or o-minimality is necessary?

Work in progress with Chris Miller.

1. Defining or not $\mathbb{Z}$

A first way to reasonably weaken o-minimality is to look at structures that are not o-minimal but do not define $\mathbb{Z}$.

S defines $\mathbb{Z}$ — no PW bound

S not o-minimal & does not define $\mathbb{Z}$:
- $(\mathbb{R}, \overline{\mathbb{Q}})$ → no PW bound
- $(\mathbb{R}, \{2^{\mathbb{Z}}\})$ → PW bound

S o-minimal — PW bound

Some test family: $f_\ell(x) = \sin(\log^\ell(x))$, $\ell \geq 1$.

$(\mathbb{R}, \Gamma_\ell)$ defines $\mathbb{Z}$ for $\ell > 1$

$(\mathbb{R}, \Gamma_1)$ does not define $\mathbb{Z}$ (C. Miller – P. Speissegger)

2. Rational points in Γ_{f_ℓ}

For given $A \in \mathbb{R}$ and $L > 0$, we first show that one can produce $P^{-1}(0)$ an algebraic curve of degree d, such that $\Gamma_{f_\ell|[A, A+L]}(\mathbb{Q}, T) \subseteq P^{-1}(0)$.
For this use Bombieri & Pila's determinant method: one can find such a curve $P^{-1}(0)$ under the hypothesis that L is small enough. Let $L = L_0$, $P = P_0 \in \mathbb{R}[X,Y]_d$.
Continue the process on $[A+L_0, A+L_0+L_1]$ for some $L_1 > 0$.
It defines a sequence of intervals $I_0 = [A, A+L_0]$, $I_1 = [A+L_0, A+L_0+L_1]$ ...
and a sequence of algebraic curves $P_0, P_1, \ldots \in \mathbb{R}[X,Y]_d$.

Statement: The number of intervals of f_ℓ one needs to cover $[1, T]$ is $\leq C(d,\ell)\, T^{\sigma(d)}$ where $\sigma(d) \xrightarrow[d \to \infty]{} 0$, as soon as the derivatives of f_ℓ are smaller and smaller. (for instance here $\left|\frac{f_\ell^{(k)}(x)}{k!}\right| \leq \frac{C(\ell,k)}{x^k}$)

Rk. The hypothesis on the derivatives is satisfied for $\zeta_{|[2;+\infty[}$ the Riemann zeta function.

Now as soon as we can bound $\# P^{-1}(0) \cap \Gamma_{f_\ell}$, with $\deg(P) = d$, one gets a bound on $\Gamma_{f_\ell}^{\text{trans}}(\mathbb{Q}, T)$.

for instance, by Khovanskii's bounds $\# P^{-1}(0) \cap \Gamma_{f_\ell} \leq 4d\ell(d+\ell+2)\log\frac{(T^2)}{\pi}$

Now fixing $d = \log(T^2)$ one finally get : $\# \Gamma_{f_\ell}^{\text{trans}}(\mathcal{L}, T) \leq C(\ell) \log^{\beta_\ell}(T^2)$ for $\beta_\ell > 0$, ($\beta_\ell = 4\ell+1$).

The same kind of bound is also obtained for $\zeta_{|]2;+\infty[}$:

$$\# \Gamma_{\zeta_{|]2;+\infty[}}(\mathcal{L}, T) \leq C . \log^2(T^2) . \log(\log(T^2))$$

(Compare to Masser, Boxall & Jones).

<u>Remark:</u> This method applies for a large class of transcendental entire functions, using bounds provided by Coman & Poletsky.
This method also applies for fast spirals (that are definable in $(\mathbb{R}, f_\ell)$).