Refined curve counting and Hrushovski-Kazhdan motivic integration

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In honor of Bernard Teissier

Aim: geometric interpretation for Block and Göttsche's tropical refined multiplicities in enumerative geometry, using Hrushovski-Kazhdan motivic integration.

Joint work with Sam Payne and Franziska Schröter.

Outline

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Motivation

Starting point: classical question in enumerative geometry.

Question

What is the number n_d of rational degree d curves through $3d-1$ general points in $\mathbb{P}^2_{\mathbb{C}}$?

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Examples: $n_1 = 1$, $n_2 = 1$, $n_3 = 12$ (later), ...

More generally: linear systems on projective toric surfaces.

Question

Let Δ be a lattice polytope in $\mathbb{R}^2$ containing $n+1$ lattice points and g interior lattice points.

What is the number of rational curves in the linear system $L(\Delta) \cong \mathbb{P}^g_{\mathbb{C}}$ of hyperplane sections through $n-g$ general points on the projective toric surface $X(\Delta)$?

We will denote this number by n_Δ.

Geometric interpretation:

Theorem (Beauville, Fantechi-Göttsche-van Straaten)

If every curve in the linear system $L(\Delta)$ is integral, then n_Δ equals the Euler characteristic of the relative compactified Jacobian of the universal curve

$$\mathcal{C} \to L(\Delta) \cong \mathbb{P}^g_{\mathbb{C}}.$$

Example

Set $\Delta = \mathrm{Conv}\{(0,0),(0,3),(3,0)\}$. Then $n = 9$, $g = 1$ and $X(\Delta) = (\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}(3))$.

Thus $n_\Delta = n_3$ and $L(\Delta)$ is the pencil of cubics in $\mathbb{P}^2_{\mathbb{C}}$ through 8 general points. This linear system has 9 base points, and blowing up these points we get an elliptic fibration

$$\mathcal{C} \to \mathbb{P}^1_{\mathbb{C}}$$

with $12 = \chi(\mathcal{C})$ singular fibers.

Combinatorial computation:

Theorem (Mikhalkin, 2005)

The invariant n_Δ is equal to the number of rational tropical curves of degree Δ through $n-g$ general points in $\mathbb{R}^2$, counted with appropriate multiplicities.

Mikhalkin's multiplicities are defined combinatorially and express how many curves in the linear system $L(\Delta)$ have the given tropicalization.

Göttsche and Shende: refinement of n_Δ to a Laurent polynomial

$$N_\Delta(y) \in \mathbb{Z}[y, y^{-1}]$$

such that

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such that

1. $N_\Delta(1) = n_\Delta$,
2. $N_\Delta(-1)$ is a similar invariant in real algebraic geometry (Welschinger).

Geometric meaning: if every curve in the linear system $L(\Delta)$ is integral, then $N_\Delta(y)$ equals the χ_y-genus of the relative compactified Jacobian of the universal curve

$$\mathcal{C} \to \mathbb{P}^g_{\mathbb{C}}.$$

Block-Göttsche: tropical computation of $N_\Delta(y)$, refining Mikhalkin's multiplicities to Laurent polynomials in y (BG-multiplicities).

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Question

What is the geometric meaning of the BG-multiplicity $BG(\Gamma)$ of an individual tropical curve Γ?

Conjecture (N.-Payne-Schröter)

If every curve in $L(\Delta)$ is integral, then the invariant $BG(\Gamma)$ is equal to the limit χ_y-genus of the relative compactified Jacobian of the locus of curves in $\mathcal{C}$ with tropicalization Γ.

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Important difficulty: χ_y is not multiplicative in smooth and proper families, knowing χ_y for the fibers is not enough.

¿Qué?

We must make sense of the phrase

"limit χ_y-genus of the relative compactified Jacobian of the locus of curves in $\mathcal{C}$ with tropicalization Γ."

Our definition is based on the theory of motivic integration of Hrushovski-Kazhdan.

Some notation

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- Tropicalization map

$$\mathrm{trop} : (K^{\times})^n \to \mathbb{R}^n : (x_1, \ldots, x_n) \mapsto (v(x_1), \ldots, v(x_n))$$

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For our problem: study locus in $L(\Delta)$ of curves with fixed tropicalization under

$$\operatorname{trop} : X(\Delta)(K) \supset (K^\times)^2 \to \mathbb{R}^2.$$

Semi-algebraic sets

Definition

A semi-algebraic subset of K^n is a finite Boolean combination of subsets of the form

$$\{x \in K^n \,|\, v(f(x)) \geq v(g(x))\}$$

where f and g are polynomials in $K[x_1, \ldots, x_n]$.

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Using affine charts, one can define semi-algebraic subsets of any algebraic variety X over K. We will often simply speak of semi-algebraic sets and leave the ambient variety X implicit.

Some examples

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2. If σ is a polyhedron in $\mathbb{R}^n$, then $\mathrm{trop}^{-1}(\sigma)$ is a semi-algebraic subset of $\mathbb{G}^n_{m,K}$.
3. If $\mathscr{X}$ is an R-scheme of finite type, then $\mathscr{X}(R)$ is a semi-algebraic subset of $\mathscr{X}_K$.

   More generally, for every locally closed subset Y of $\mathscr{X}_k$,

$$\mathrm{sp}_{\mathscr{X}}^{-1}(Y) = \{x \in \mathscr{X}(R) \,|\, x_k \in Y\}$$

   is semi-algebraic.

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   is semi-algebraic.
4. The locus of curves in $L(\Delta) \cong \mathbb{P}^g_K$ with fixed tropicalization is semi-algebraic [Katz].

The Grothendieck ring of semi-algebraic sets

A morphism of semi-algebraic sets is a map whose graph is semi-algebraic. The image (resp. inverse image) of a semi-algebraic set under a semi-algebraic morphism is semi-algebraic (Robinson's QE for ACVF).

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VF_K: category of semi-algebraic sets.

$K_0(\mathrm{VF}_K)$: Grothendieck ring, with usual scissor relations:

$$[S]+[T]=[S\cup T]+[S\cap T]$$

if S, T are semi-algebraic subsets of some K-variety X.

To construct interesting invariants (e.g. "limit χ_y-genus") of semi-algebraic sets, we will make use of the motivic volume

$$\mathrm{Vol} : K_0(\mathrm{VF}_K) \to K_0(\mathrm{Var}_k)$$

of Hrushovski-Kazhdan.

The motivic volume of Hrushovski-Kazhdan

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We will describe two natural constructions to produce classes in $K_0(\mathrm{VF}_K)$ from objects that live over the value group or the residue field.

First construction

Let n be a non-negative integer and let σ be a real polyhedron of dimension at most n. We embed σ in $\mathbb{R}^n$ and we denote by $\Theta(\sigma, n)$ the class of

$$\mathrm{trop}^{-1}(\sigma) \subset (K^*)^n$$

in $K_0(\mathrm{VF}_K)$. It does not depend on the chosen embedding.

Second construction

Let n be a non-negative integer and let Y be a k-variety of dimension at most n. Then we can decompose Y into locally closed subsets U such that there exists a connected smooth R-scheme $\mathscr{X}$ of relative dimension n and an immersion $U \to \mathscr{X}_k$.

Second construction

Let n be a non-negative integer and let Y be a k-variety of dimension at most n. Then we can decompose Y into locally closed subsets U such that there exists a connected smooth R-scheme $\mathscr{X}$ of relative dimension n and an immersion $U \to \mathscr{X}_k$.

We set

$$\Theta(U, n) = [\mathrm{sp}_{\mathscr{X}}^{-1}(U)] \quad \in K_0(\mathrm{VF}_K)$$

and we define $\Theta(Y, n)$ additively. This definition is independent of all choices (because R is henselian).

These constructions are not completely orthogonal: denoting by Δ_0 the 0-simplex (i.e., a point), we have

$$\Theta(\Delta_0, 1) = \Theta(\mathbb{G}_{m,k}, 1) = [R^\times]$$

in $K_0(\mathrm{VF}_K)$.

We now consider the graded rings

$$K_0(\mathbb{R}[*]) = \bigoplus_{n\geq 0} K_0(\mathbb{R}[n]) \quad \text{and} \quad K_0(\mathrm{Var}_k[*]) = \bigoplus_{n\geq 0} K_0(\mathrm{Var}_k[n])$$

where the summands are the Grothendieck groups of real polyhedra, resp. k-varieties, of dimension $\leq n$.

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We view these graded rings as $\mathbb{Z}[\tau]$ algebras by sending τ to

$$[\Delta_0]_1 \in K_0(\mathbb{R}[1]) \quad \text{and} \quad [\mathbb{G}_{m,k}]_1 \in K_0(\mathrm{Var}_k[1]),$$

respectively.

Remarkable results (Hrushovski-Kazhdan, 2006):

1. The morphism

$$\Theta : K_0(\mathbb{R}[*]) \otimes_{\mathbb{Z}[\tau]} K_0(\mathrm{Var}_k[*]) \to K_0(\mathrm{VF}_K)$$

is surjective.

Remarkable results (Hrushovski-Kazhdan, 2006):

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   is surjective.

2. We can explicitly describe its kernel, denoted I.

Description of the kernel

The class in $K_0(\mathrm{VF}_K)$ of the open unit disc

$$D = \{x \in K \mid v(x) > 0\}$$

can be written in two different ways:

1. $[D] = [D \setminus \{0\}] + [\mathrm{Spec}\, K] = \Theta(\mathbb{R}_{>0}, 1) + \Theta(\mathrm{Spec}\, k, 0),$

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Thus $[\mathbb{R}_{>0}]_1 + [\mathrm{Spec}\, k]_0 - [\mathrm{Spec}\, k]_1$ lies in I. The striking fact is that it even generates I.

By inverting Θ, we obtain a ring isomorphism

$$K_0(\mathrm{VF}_K) \to \left(K_0(\mathbb{R}[*]) \otimes_{\mathbb{Z}[\tau]} K_0(\mathrm{Var}_k[*])\right)/I.$$

We will now use it to construct a ring morphism

$$\mathrm{Vol} : K_0(\mathrm{VF}_K) \to K_0(\mathrm{Var}_k)$$

that we call the motivic volume.

There is an obvious ring morphism

$$K_0(\mathrm{Var}_k[*]) \to K_0(\mathrm{Var}_k)$$

that forgets the grading.

We can also define a ring morphism

$$K_0(\mathbb{R}[*]) \to K_0(\mathrm{Var}_k) : [\sigma]_n \mapsto \chi'(\sigma)(\mathbb{L}-1)^n$$

where $\mathbb{L} = [\mathbb{A}^1_k]$ and

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The invariant χ' is fully characterized by the property that it is additive and $\chi'(\sigma) = 1$ for every closed polyhedron σ.

These morphisms send $[\mathrm{Spec}\, k]_n$ to 1 and $[\mathbb{R}_{>0}]_n$ to zero for all $n \geq 0$. Thus they induce a ring morphism

$$\mathrm{Vol} : K_0(\mathrm{VF}_K) \cong \left(K_0(\mathbb{R}[*]) \otimes_{\mathbb{Z}[\tau]} K_0(\mathrm{Var}_k[*])\right) / I \to K_0(\mathrm{Var}_k).$$

Example

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2. If σ is a polyhedron in $\mathbb{R}^n$ then

$$\mathrm{Vol}(\mathrm{trop}^{-1}(\sigma)) = \chi'(\sigma)(\mathbb{L}-1)^n.$$

Definition

The limit χ_y-genus of a semi-algebraic set is defined by composing

$$\mathrm{Vol} : K_0(\mathrm{VF}_K) \to K_0(\mathrm{Var}_k)$$

with the χ_y-realization

$$\chi_y : K_0(\mathrm{Var}_k) \to \mathbb{Z}[y, y^{-1}].$$

Computing the motivic volume

Definition

A strictly semi-stable R-scheme is an R-scheme $\mathscr{X}$ of finite type that admits locally an étale morphism to a scheme of the form

$$S_{n,r,a} = \operatorname{Spec} R[x_0, \ldots, x_n]/(x_0 \cdot \ldots \cdot x_r - a)$$

with $r \leq n$ and $a \in \mathfrak{m} \setminus \{0\}$.

Note that

$$\begin{aligned} \mathrm{sp}_{S_{n,n,a}}^{-1}(O) &= \{x \in \mathfrak{m}^{n+1} \mid x_0 \cdot \ldots \cdot x_n = a\} \\ &\cong \mathrm{trop}^{-1}(\Delta^o_{n,a}) \subset (K^\times)^n \end{aligned}$$

with

$$\Delta^o_{n,a} = \{(t_1, \ldots, t_n) \in \mathbb{R}^n_{>0} \mid t_1 + \ldots + t_n < v(a)\}.$$

Note that

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with

$$\Delta_{n,a}^o = \{(t_1, \ldots, t_n) \in \mathbb{R}_{>0}^n \mid t_1 + \ldots + t_n < v(a)\}.$$

Since $\chi'(\Delta_{n,a}^o) = (-1)^n$, it follows that

$$\mathrm{Vol}(\mathrm{sp}_{S_{n,n,a}}^{-1}(O)) = (1 - \mathbb{L})^n.$$

Writing

$$\mathscr{X}_k = \sum_{i \in I} E_i,$$

a slight generalization of this computation yields the familiar formula

$$\mathrm{Vol}(\mathscr{X}(R)) = \sum_{\emptyset \neq J \subset I} (1 - \mathbb{L})^{|J|-1} [E_J^o] \quad \in K_0(\mathrm{Var}_k).$$

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Corollary (NPS)

Assume that an embedding of $k((t))$ in K is given. For every generically smooth $k[[t]]$-variety $\mathscr{X}$, the image of $\mathrm{Vol}(\mathscr{X}(R))$ in $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ coincides with Denef and Loeser's motivic nearby fiber of $\mathscr{X}$ (forgetting the monodromy).

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We will only give a sample formula, which generalizes [Katz - Stapledon] (with a more direct proof).

Theorem (NPS)

Let X be a schön subvariety of $\mathbb{G}_{m,K}^n$ of dimension d and let Σ be a tropical polyhedral decomposition of $\mathrm{trop}(X)$. Then we have

$$[X] = \sum_{\sigma \in \Sigma} \Theta\left([Y(\sigma)]_{d-\dim(\sigma)} \otimes [\sigma]_{\dim(\sigma)}\right) \quad \in K_0(\mathrm{VF}).$$

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It follows that

$$\mathrm{Vol}(X) = \sum_{\sigma \in \Sigma^b} (-1)^{\dim(\sigma)} [\mathrm{in}_\sigma(X)]$$

where the sum is taken over the bounded cells σ of Σ. Moreover, there exists a similar formula for the tropical compactification of X.