Arc spaces and some adjacency problems of plane curves.

María Pe Pereira

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Joint work in progress with
Javier Fernández de Bobadilla and Patrick Popescu-Pampu
Arcspace of \((\mathbb{C}^2, 0)\).

Arc (through the origin) of \(\mathbb{C}^2\): germ of parametrization through the origin:

\[
\gamma : (\mathbb{C}, 0) \rightarrow (X, O) \subset (\mathbb{C}^2, O)
\]

\[
t \mapsto (\sum_i a_i^1 t^i, ..., \sum_i a_i^n t^i)
\]

\[
0 \mapsto O
\]

Formal arcs are considered: the power series may not converge.
It is an infinite affine space.
It is irreducible.
Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A divisor is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$. 

\[
\text{Composition of Blow-ups} \\
\pi \\
\mathbb{C}^2 \quad \tilde{X}
\]

$E_1$, $E_2$, $E_3$
Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is an exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$. 

![Diagram showing Nash sets associated to divisors over $O \in \mathbb{C}^2$.](image)
Nash sets associated to divisors over $O \in \mathbb{C}^2$.

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Take the minimal model for $E_i$.

$$N_i = \{ \gamma : \tilde{\gamma}(0) \in E_i \}$$

The *Nash set* is its closure $\overline{N}_i$. 

$$\begin{array}{c}
\mathbb{C}^2 \\
\mathbb{C} \\
\gamma \\
\pi \\
\text{Composition of Blow-ups} \\
\tilde{\gamma} \\
\text{Im} \gamma \\
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The Nash set is its closure $\overline{N_i}$.

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- They are cylindrical: they are determined in order $k$. 
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- $N_{E_0}$ is equal to the whole arc space.
- Nash sets are irreducible.
- They are cylindrical: they are determined in order $k$.
- They have finite codimension.
- They are all different: $\overline{N}_i \neq \overline{N}_j$. 
Nash sets.

Take a composition of blow ups in point above the origin.

Take the minimal model for $E_i$.

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The Nash set is its closure $\overline{N}_i$.

- What is the closure $\overline{N}_{E_i}$?
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- What is the closure $\overline{N_{E_i}}$?
  
  If there is a family as for example
  
  \[ \alpha_s(t) = (t^5 + st^3; t^4 + st^4) \]

  with $\alpha_s \in N_F$ and $\alpha_0 \in N_E$, then $\alpha_0 \in \overline{N_F}$. 

\( \gamma \)

\( \text{Im} \gamma \)

\( \mathbb{C} \)

\( \mathbb{C}^2 \)

Composition of Blow-ups

\( \pi \)

\( \hat{\gamma} \)

\( E_1, E_2, E_3 \)

\( \hat{\mathcal{X}} \)

\( \mu \)

\( \text{Im} \hat{\gamma} \)

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The Nash set is its closure \( \overline{N_i} \).

- What is the closure \( \overline{N_{E_i}} \)?

If there is a family as for example

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\alpha_s(t) = (t^5 + st^3; t^4 + st^4)
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with \( \alpha_s \in N_F \) and \( \alpha_0 \in N_E \), then \( \alpha_0 \in \overline{N_F} \).

- GENERALISED NASH PROBLEM: Determine when \( \overline{N_E} \subset \overline{N_F} \).
Nash problem.

For a singular variety \((X, \text{Sing } X)\), the components of the space of arcs centered at \(\text{Sing } X\) are of the form \(\overline{N}_E\) for certain exceptional components \(E\) of a resolution of singularities. These components appear in any resolution.
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- **Surface singularities** (Nash Conjecture, Theorem 2011, J. Fernandez de Bobadilla, M. P. P.): The components of the arcspace are in bijection with exceptional components of the minimal resolution.

- **Higher dimensional case** (partial result 2014, T. de Fernex, R. Docampo). Components in terminal models give components of the space of arcs. But there are more... still open.
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If some \(\overline{N}_E\) isn’t a component of the space of arcs, then \(\overline{N}_E \subseteq \overline{N}_F\) for some \(F\).
Generalised Nash Problem: describe the inclusions/adjacencies $\overline{N}_E \subset \overline{N}_F$. 
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Trivial inclusions: $F < E$ (E dominates F) implies $N_E \subset \overline{N}_F$
How to check if $\overline{N}_E \subset \overline{N}_F$?

**Theorem** (Fernández de Bobadilla, 2009)

*Given two divisors above $O \in \mathbb{C}^2$, the following are equivalent:*

1. $\overline{N}_E \subset \overline{N}_F$
2. there exists a convergent family of convergent arcs $\alpha$ realising the inclusion with $\alpha_0 \in \dot{N}_E$ and $\alpha_s \in N_F$.
3. for any convergent arc $\gamma \in \dot{N}_E$ there exists a family of convergent arcs $\alpha$ realising the inclusion with $\alpha_0 = \gamma$ (and $\alpha_s \in N_F$).
$\bar{N}_E \subset \bar{N}_F$ doesn’t imply $F < E$.

- $(t^5 + s^3t^3, (1 + s^4)t^4)$, two different tangents for $s = 0$ and $s \neq 0$. 

\[ s=0 \quad s \neq 0 \]
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- $(t^5 + s^3 t^3, (1 + s^4) t^4)$, two different tangents for $s = 0$ and $s \neq 0$.

- Multiple examples:
We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

A divisor is determined by the combinatorics + moduli...
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.
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Observe: not all the divisors are the final one for the minimal embedded resolution of a branch, but only the blow ups of satellite points.
We want to describe $\text{NE} \subset \text{NF}$ (partial order among divisors over $O \in \mathbb{C}^2$)

Take into account the contact order between $E$ and $F$, and much more (moduli for free points also count apriori!)...
We want to describe $\overline{\mathcal{N}}_E \subset \overline{\mathcal{N}}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

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Talk about combinatorics of the pair $(E, F)$. We write $(E, F) \equiv (E', F')$...
We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

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Talk about combinatorics of the pair $(E, F)$. We write $(E, F) \equiv (E', F')$...
Domination relation $F < E = \text{Inclusion of Enriques diagrams.}$
Divisorial valuation $ord_E = \text{vanishing order along the divisor } E$.

Can be computed intersecting with appropriate arcs $\gamma$ in $N_E$:

$$ord_E(f) = I_O(f, \gamma) = ord_t(f \circ \gamma(t)).$$

Choosing a family of arcs with an appropriate $\alpha_0 \in N_E$ and $\alpha_s \in N_F$ we get

$$ord_F(h) \leq ord_t(h \circ \alpha_s(t)) \leq ord_t(h \circ \alpha_0(t)) = ord_E(h)$$

for all $h \in \mathbb{C}[[x, y]]$. 
Valuative criterion in arc spaces (A. Reguera, C. Plénat, S. Ishii...)

C. Plénat in general in Annal Inst. Fourier 2005:
- $\overline{N}_E \subset \overline{N}_F$ implies $ord_E \leq ord_F$

S. Ishii in *Maximal Divisorial Sets*:
- if $F$ is toric, also the converse is true: $ord_E \leq ord_F$ implies $\overline{N}_E \subset \overline{N}_F$.
- She found a counterexample for the converse in general ($ord_F \leq ord_E$ but $\overline{N}_E \not\subseteq \overline{N}_F$).

![Diagram](https://via.placeholder.com/150)
\( F < E \implies \overline{N}_E \subseteq \overline{N}_F \implies \text{ord}_F \leq \text{ord}_E \)
Other tools to rule out adjacencies?

- $\overline{N}_E \subsetneq \overline{N}_F$ implies $\text{codim}(\overline{N}_F) < \text{codim}(\overline{N}_E)$.

In [de Fernex, Ein, Ishii, Lazarsfeld, Mustata’200?]:

$$\text{codim}(\overline{N}_E) = 1 + \text{disc}(E, \mathbb{C}^2).$$

The discrepancy of $E$ is the coefficient of $E$ in $K_X/\mathbb{C}^2$ where $\pi : X \to \mathbb{C}^2$ is any model where $E$ appears.

- It is not a sufficient criterium (even with toric examples)
- Neither plus the valuative criterium (counterexample of Ishii with the same discrepancy).

The problem turns very difficult...
Example of topological types.

- $ord_F \leq ord_E$
- $disc(E) + 1 = \text{codim}(\overline{N}_E) = 21 > \text{codim}(\overline{N}_F) = disc(F) + 1 = 17$
- $\overline{N}_E \nsubseteq \overline{N}_F$
Main result about inclusions

Recall that $\overline{N}_E \subset \overline{N}_F$ depends on the relative position of $E$ and $F$, but ... we don't know a priori that it is a combinatorial problem, it also depends on the moduli of the free points!.

**Theorem**

*Assume there exists a wedge $\alpha$ realising the adjacency $\overline{N}_E \subset \overline{N}_F$. If $(E', F') \equiv (E, F)$ then there exists a wedge realising the adjacency $\overline{N}_{E'} \subset \overline{N}_{F'}$.***
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**Corollary**

Assume we have that \( \overline{N}_E \subset \overline{N}_F \). Let \( i_0 \) be the contact order between \( E \) and \( F \). Then, we have that

\[
\bigcup_{E' \equiv \geq i_0 E} \overline{N}_{E'} \subset \bigcap_{F' \equiv \geq i_0 F} \overline{N}_{F'}
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where \( A \equiv \geq i_0 B \) means that \( A \) has the same combinatorics as \( B \) and their contact order is \( \geq i_0 \).
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We improve the log-discrepancy inequality in many cases.
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We improve the log-discrepancy inequality in many cases. Other conjectures...
\[(E, F) \equiv (E', F') \Rightarrow [\overline{N}_E \subset \overline{N}_F \iff \overline{N}_{E'} \subset \overline{N}_{F'}]\]
\[(E, F) \equiv (E', F') \Rightarrow [\overline{N}_E \subsetneq \overline{N}_F \iff \overline{N}_{E'} \subsetneq \overline{N}_{F'}]\]

To change the complex structure we can use:

- Let \(g : X \rightarrow Y\) is a non-ramified covering of differentiable manifolds. A complex structure on \(Y\) can be lifted to \(X\) so that \(g\) is a local biholomorphism.
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- (Grauert-Remmert) Let \(A\) be a normal analytic space, let \(B \subset A\) be a closed analytic subset such that \(A \setminus B\) is dense in \(A\). Let

\[f : U \to A \setminus B\]

be a finite and étale analytic morphism. Then there exists a finite analytic extension

\[\tilde{f} : V \to A\]

from a normal analytic space \(V\). Moreover \(V\) is unique up to isomorphism.
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We can assume the wedge \(\alpha : \mathbb{C}^2 \to \mathbb{C}^2\) is algebraic, that is there exists polynomials \(F_1, F_2 \in \mathbb{C}[s, t, x, y]\) such that

\[
    F_1(s, t, \alpha_1(s, t)) = F_2(s, t, \alpha_2(s, t)) = 0.
\]
Coming back to the valuative criterium...

Recall: $\overline{N}_E \subset \overline{N}_F$ implies there exists a family of parametrizations $\alpha(t, s)$ with $\alpha_0(t) \in \dot{N}_E$ and $\alpha_s \in \dot{N}_F$ for all $s \in \Lambda \setminus \{0\}$.

- Deforming a little $\alpha$, we can assume that
  \[
  \alpha^{-1}(O) = \{0\} \times \Lambda.
  \]

- The equation $F(x, y, s)$ of
  \[
  \text{Im}[(t, s) \mapsto (\alpha(t, s), s) \in \mathbb{C}^2 \times \Lambda]
  \]
  gives a deformation of plane curves given by $f_s(x, y) := F(x, y, s)$ where $f_0(x, y) = 0$ lifts transversally to $E$ and all $f_s(x, y) = 0$ lift transversally to $F$ for all $s \neq 0$.

- These deformations have a special property: for $s \neq 0$ they can be resolved simultaneously by a sequence of blow ups, they fix the free points (for $F$).
Valuative criterium.

- Let $f_s$ be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some $E$ and $f_s = 0$ have strict transforms transverse to a fixed $F$ for all $s \neq 0$, then

$$ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x, y]].$$

We have $I_O(h, f_s) \leq I_O(h, f_0)$ but is not enough...
Valuative criterium.

Let $f_s$ be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some $E$ and $f_s = 0$ have strict transforms transverse to a fixed $F$ for all $s \neq 0$, then

$$\text{ord}_F(h) \leq \text{ord}_E(h) \quad \forall h \in \mathbb{C}[[x, y]].$$

We have $I_O(h, f_s) \leq I_O(h, f_0)$ but is not enough...

Proof

Take embedded resolution $(\tilde{X}, D = \bigcup_i D_i) \to (\mathbb{C}^2, O)$ of $f_s = 0$ and $f_0 = 0$.

Look at it in family $\tilde{X} \times \Lambda \to \mathbb{C}^2 \times \Lambda$.

Let $Y$ be the strict transform of $F = 0$ ($F(x, y, s) := f_s(x, y)$). Observe

$$Y_s = \{f_s = 0\} \quad \text{for } s \neq 0$$

$$Y_0 = \{f_0 = 0\} + \sum_{k} d_k D_k, \quad \text{with } d_k \geq 0.$$

We get $Y_0 \cdot D_i = Y_s \cdot D_i$ for any $i$. Putting $M = (D_i \cdot D_j)$, ($E = D_0$, $F = D_n$),

$$(1, 0, \ldots, 0)^t + M(d_1, \ldots, d_n)^t = (0, \ldots, 0, 1)^t.$$

$$-M^{-1}(1, 0, \ldots, 0, -1)^t = (d_1, \ldots, d_n)^t \geq 0.$$

and the entries of $-M^{-1}$ are exactly $\text{ord}_{D_i}(h_{D_i}) = I_O(h_{D_i}, h_{D_i})$. 
Valuative criterium.

Let \( f_s \) be a deformation fixing the free points. If \( f_0 = 0 \) has strict transform transverse to some \( E \) and \( f_s = 0 \) have strict transforms transverse to a fixed \( F \) for all \( s \neq 0 \), then

\[
ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x, y]].
\]

We have \( I_O(h, f_s) \leq I_O(h, f_0) \) but is not enough...

Reciprocally, if

\[
ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x, y]],
\]

then, taking \( h_E = 0 \) and \( h_F = 0 \) with strict transform transverse to \( E \) and \( F \) in a model of \( E + F \), then

\[
h_E + s \cdot h_F
\]

have strict transform transverse to \( F \) for \( s \neq 0 \) small enough. (Also proved by M. Alberich y J. Roe).
Valuative criterium.

Let $f_s$ be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some $E$ and $f_s = 0$ have strict transforms transverse to a fixed $F$ for all $s \neq 0$, then

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$$h_E + s \cdot h_F$$

have strict transform transverse to $F$ for $s \neq 0$ small enough. (Also proved by M. Alberich y J. Roe).

Proof

*Check it works.*
Valuative criterium.

Let $f_s$ be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some $E$ and $f_s = 0$ have strict transforms transverse to a fixed $F$ for all $s \neq 0$, then

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Summarizing:

**Proposition**

Let $E$ and $F$ be two prime divisors. There exists a deformation $f_s$ of a curve $f_0 = 0$ that lifts transversal to $E$ that fixes the free points for $F$ ($f_s = 0$ has strict transform transverse to $F$ for $s \neq 0$) if and only if $\text{ord}_F \leq \text{ord}_E$.

So, the valuative criterion is a criterion for the existence of deformations of functions not of parametrizations!
Adjacency problems.

- **CLASSICAL ONE**: Given two topological types $f = 0$ and $g = 0$ in $(\mathbb{C}^2, 0)$, study when there exists a deformation $f_t = 0$ where $f_0 = 0$ has the topological type of $f = 0$ and $f_t = 0$ the one of $g = 0$.

- **OUR OBSERVATION**: deformation fixing the free points of the generic curves are characterized by the valuative criterium.
Valuative criterium.

Proposition

Let $E$ and $F$ be prime divisors over $O \in \mathbb{C}^2$. There exists a deformation $f_s$ of a curve $f_0 = 0$ that lifts transversal to $E$ that fixes the free points for $F$ ($f_s = 0$ has strict transform transverse to $F$ for $s \neq 0$) if and only if $\text{ord}_F \leq \text{ord}_E$.

Good things about the result and our problem:

- It talks about concrete divisors, not only topological types.
- Takes into account the contact order of $E$ and $F$.
- They are very easy to check finite conditions (inequalities for $h_D$ with $D$ in the minimal model of $F$) $\Rightarrow$ Algorithm!
- Also works for $F$ a non-prime divisor: if $F = \sum_i a_i F_i$ then we the condition is $\text{ord}_F := \sum_i a_i \text{ord}_{F_i} \leq \text{ord}_E$.

Bad news:

- Not all the adjacencies are of this type.
We recover many of the adjacencies from Arnol'd’s list.

Only 7 out of the 93 classical adjacencies between simple singularities of $\mu \leq 8$ are not realizable.
We recover many of the adjacencies from Arnol’d’s list.

For example, $\text{ord}_{A_5} \not\preceq \text{ord}_{E_6}$ but still there exists a deformation

$$y^3 + x^4 + s^2 y^2 + 2sx^2 y.$$
We recover many of the adjacencies from Arnol’d’s list.

Some were not in Arnol’d’s list:

\[ Z_{11} = S_{2,4,5} \rightarrow E_8, \quad Z_{12} = S_{2,4,6} \rightarrow J_{10} = T_{2,3,6}, \quad W_{17} \rightarrow Z_{13} = S_{2,4,7} \]

Some are not realizable:

\[ W_{18} \leftrightarrow Z_{17}, \quad Z_{11} \leftrightarrow J_{10}, \quad X_9 \leftrightarrow E_7. \]
Relation to the study of $\delta$ constant stratum.

- Recall Teissier’s Theorem: a deformation $f_t$ admits a parametrization in family if and only if it is $\delta$-constant. 
  $(\delta(C,0) = \dim_{\mathbb{C}}(\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,0}))$.

- Describe all the $\overline{N}_E \subset \overline{N}_F$ is equivalent to describe which of the deformations fixing the free points are in the $\delta$-constant stratum.

- Our problem is slightly different to the classical study of the $\delta$-constant stratum: may be easier?
Happy birthday and thank you!
Order and duality for topological types that are resolved in $n$ blow-ups.
Order and duality for topological types that are resolved in $n$ blow-ups.

Take combinatorial information ($1/0$) about $n - 3$ edges (straight/curve) and $n - 3$ vertices (broken between straight/smooth). Combinatorics induces a partial order: the more straight lines and broken vertices, the bigger.

You get a duality that inverting the partial order just interchanging broken/curve and smooth/straight and reading backwards.
Order and duality for topological types that are resolved in $n$ blow-ups.
Order and duality for topological types that are resolved in $n$ blow-ups.
Order and duality for topological types that are resolved in $n$ blow-ups.

It is just a combinatorial happening for the moment, will it appear in a deeper context?