Arc spaces and some adjacency problems of plane curves.

María Pe Pereira

ICMAT, Madrid

23 de junio de 2015

Joint work in progress with
Javier Fernández de Bobadilla and Patrick Popescu-Pampu

Arcspace of $(\mathbb{C}^2, 0)$.

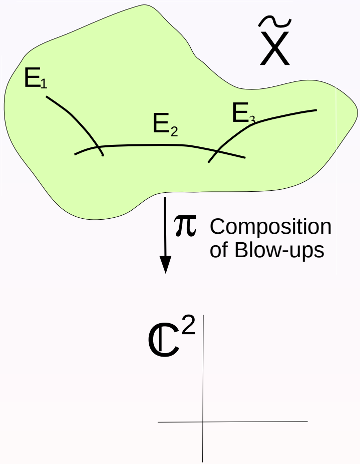
Arc (through the origin) of $\mathbb{C}^2$: germ of parametrization through the origin:

$$\begin{array}{rrcl} \gamma : & (\mathbb{C}, 0) & \longrightarrow & (X, O) \subset (\mathbb{C}^2, O) \\ & t & \longmapsto & (\sum_i a_i^1 t^i, \ldots, \sum_i a_i^n t^i) \\ & 0 & \longmapsto & O \end{array}$$

Formal arcs are considered: the power series may not converge.
It is an infinite affine space.
It is irreducible.

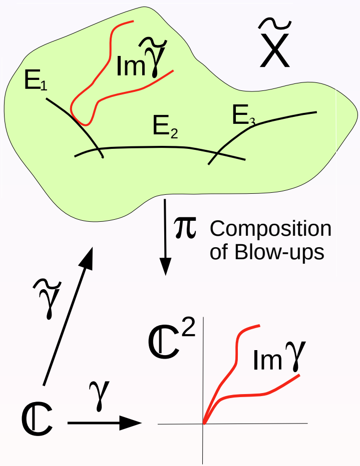
Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$.



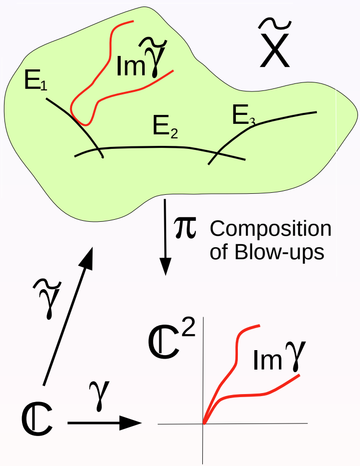
Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$.



Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$.



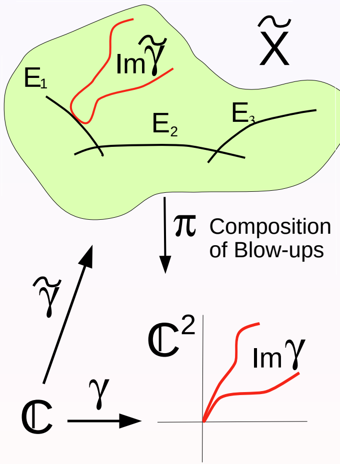
Take the minimal model for E_i.

$$N_i = \{\gamma : \widetilde{\gamma}(0) \in E_i\}$$

The *Nash set* is its closure $\overline{N}_i$.

Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$.



Take the minimal model for E_i.

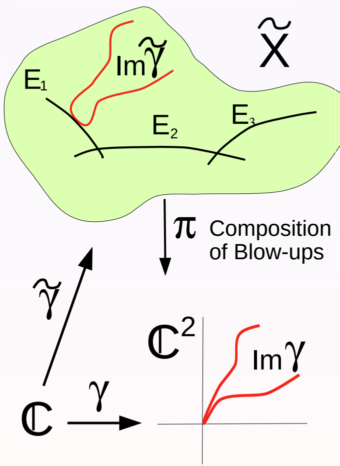
$$N_i = \{\gamma : \tilde{\gamma}(0) \in E_i\}$$

The *Nash set* is its closure $\overline{N}_i$.

- N_{E_0} is equal to the whole arc space.

Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$.



Take the minimal model for E_i.

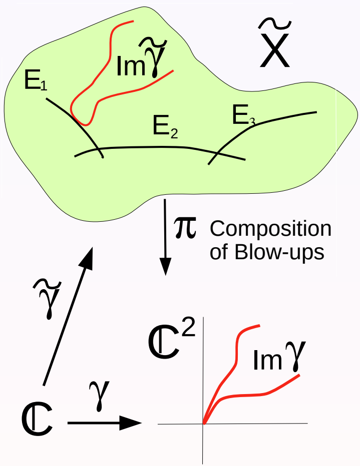
$$N_i = \{\gamma : \tilde{\gamma}(0) \in E_i\}$$

The *Nash set* is its closure $\overline{N}_i$.

- N_{E_0} is equal to the whole arc space.
- Nash sets are irreducible.

Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$.



Take the minimal model for E_i.

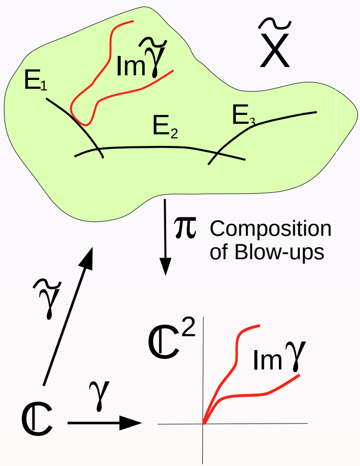
$$N_i = \{\gamma : \tilde{\gamma}(0) \in E_i\}$$

The *Nash set* is its closure $\overline{N}_i$.

- N_{E_0} is equal to the whole arc space.
- Nash sets are irreducible.
- They are cylindrical: they are determined in order k.

Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$.



Take the minimal model for E_i.

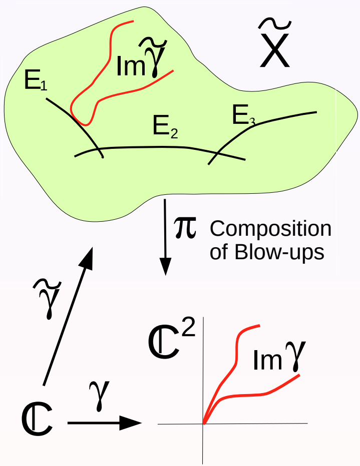
$$N_i = \{\gamma : \tilde{\gamma}(0) \in E_i\}$$

The *Nash set* is its closure $\overline{N}_i$.

- N_{E_0} is equal to the whole arc space.
- Nash sets are irreducible.
- They are cylindrical: they are determined in order k.
- They have finite codimension.

Nash sets associated to divisors over $O \in \mathbb{C}^2$.

A *divisor* is a exceptional component of a composition of blow ups in points above $O \in \mathbb{C}^2$.



Take the minimal model for E_i.

$$N_i = \{\gamma : \tilde{\gamma}(0) \in E_i\}$$

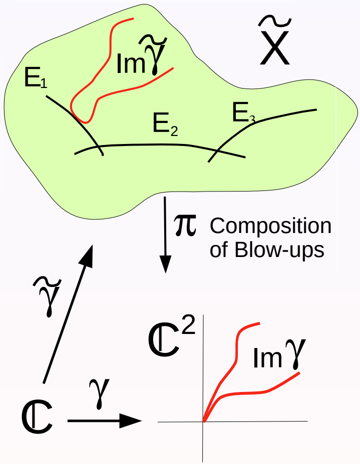
The *Nash set* is its closure $\overline{N}_i$.

- N_{E_0} is equal to the whole arc space.
- Nash sets are irreducible.
- They are cylindrical: they are determined in order k.
- They have finite codimension.
- They are all different: $\overline{N}_i \neq \overline{N}_j$.

Nash sets.

Take a composition of blow ups in point above the origin.

Take the minimal model for E_i.



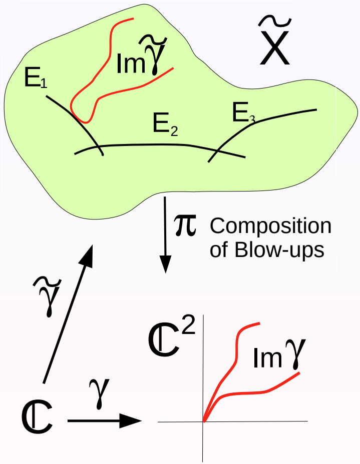
$$N_i = \{\gamma : \tilde{\gamma}(0) \in E_i\}$$

The Nash set is its closure $\overline{N}_i$.

- What is the closure $\overline{N}_{E_i}$?

Nash sets.

Take a composition of blow ups in point above the origin.



Take the minimal model for E_i.

$$N_i = \{\gamma : \tilde{\gamma}(0) \in E_i\}$$

The Nash set is its closure $\overline{N}_i$.

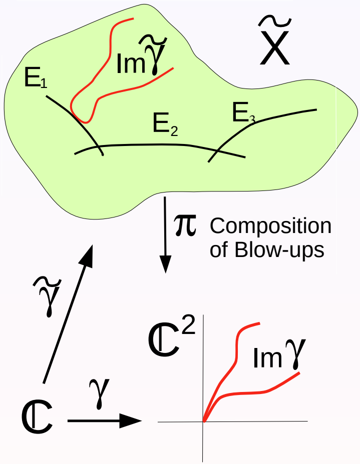
- What is the closure $\overline{N}_{E_i}$?
  If there is a family as for example

$$\alpha_s(t) = (t^5 + st^3; t^4 + st^4)$$

with $\alpha_s \in N_F$ and $\alpha_0 \in N_E$, then $\alpha_0 \in \overline{N}_F$.

Nash sets.

Take a composition of blow ups in point above the origin.



Take the minimal model for E_i.

$$N_i = \{\gamma : \tilde{\gamma}(0) \in E_i\}$$

The Nash set is its closure $\overline{N}_i$.

- What is the closure $\overline{N}_{E_i}$?
  If there is a family as for example

  $$\alpha_s(t) = (t^5 + st^3; t^4 + st^4)$$

  with $\alpha_s \in N_F$ and $\alpha_0 \in N_E$, then $\alpha_0 \in \overline{N}_F$.
- GENERALISED NASH PROBLEM: Determine when $\overline{N}_E \subset \overline{N}_F$.

Nash problem.

For a singular variety $(X, Sing\ X)$, the components of the space of arcs centered at $Sing\ X$ are of the form $\overline{N}_E$ for certain exceptional components E of a resolution of singularities. These components appear in any resolution.

Nash problem.

For a singular variety $(X, Sing\ X)$, the components of the space of arcs centered at $Sing\ X$ are of the form $\overline{N}_E$ for certain exceptional components E of a resolution of singularities. These components appear in any resolution.

- **Surface singularities** (Nash Conjecture, Theorem 2011, J. Fernandez de Bobadilla, M. P. P.): The components of the arcspace are in bijection with exceptional components of the minimal resolution.
- **Higher dimensional case** (partial result 2014, T. de Fernex, R. Docampo). Components in terminal models give components of the space of arcs. But there are more... still open.

Nash problem.

For a singular variety $(X, Sing\ X)$, the components of the space of arcs centered at $Sing\ X$ are of the form $\overline{N}_E$ for certain exceptional components E of a resolution of singularities. These components appear in any resolution.

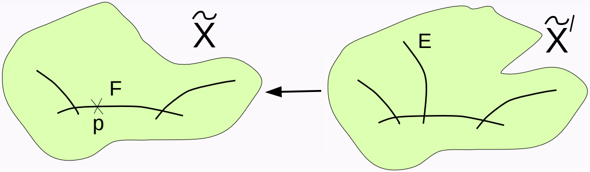
- **Surface singularities** (Nash Conjecture, Theorem 2011, J. Fernandez de Bobadilla, M. P. P.): The components of the arcspace are in bijection with exceptional components of the minimal resolution.
- **Higher dimensional case** (partial result 2014, T. de Fernex, R. Docampo). Components in terminal models give components of the space of arcs. But there are more... still open.

If some $\overline{N}_E$ isn't a component of the space of arcs, then $\overline{N}_E \subseteq \overline{N}_F$ for some F.

Generalised Nash Problem: describe the inclusions/adjacencies $\overline{N}_E \subset \overline{N}_F$.

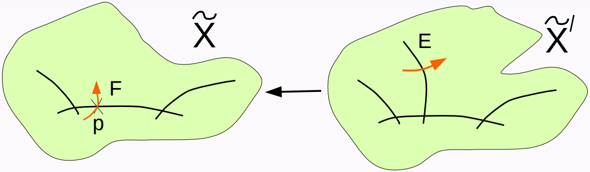
Generalised Nash Problem: describe the inclusions/adjacencies $\overline{N}_E \subset \overline{N}_F$.

Trivial inclusions: $F < E$ (E dominates F) implies $N_E \subset \overline{N}_F$



Generalised Nash Problem: describe the inclusions/adjacencies $\overline{N}_E \subset \overline{N}_F$.

Trivial inclusions: $F < E$ (E dominates F) implies $N_E \subset \overline{N}_F$



How to check if $\overline{N}_E \subset \overline{N}_F$?

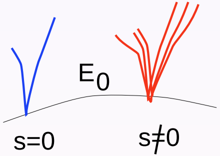
Theorem (Fernández de Bobadilla, 2009)

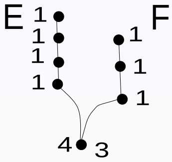
Given two divisors above $O \in \mathbb{C}^2$, the following are equivalent:

1. $\overline{N}_E \subset \overline{N}_F$
2. *there exists a convergent family of convergent arcs α realising the inclusion with $\alpha_0 \in \dot{N}_E$ and $\alpha_s \in N_F$.*
3. *for any convergent arc $\gamma \in \dot{N}_E$ there exists a family of convergent arcs α realising the inclusion with $\alpha_0 = \gamma$ (and $\alpha_s \in N_F$).*

$\overline{N}_E \subset \overline{N}_F$ doesn't imply $F < E$.

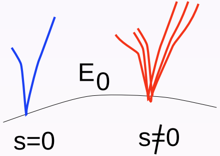
- $(t^5 + s^3t^3, (1+s^4)t^4)$, two different tangents for $s = 0$ and $s \neq 0$.

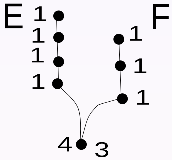




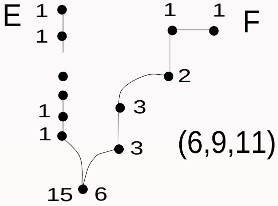
$\overline{N}_E \subset \overline{N}_F$ doesn't imply $F < E$.

- $(t^5 + s^3t^3, (1+s^4)t^4)$, two different tangents for $s = 0$ and $s \neq 0$.



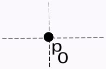


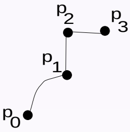
- Multiple examples:



We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

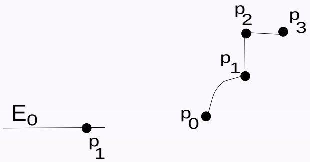
A divisor is determined by the combinatorics + moduli...
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.





We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

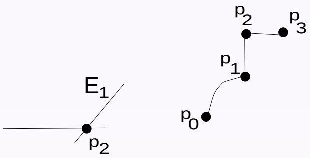
A divisor is determined by the combinatorics + moduli...
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.



We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

A divisor is determined by the combinatorics + moduli...

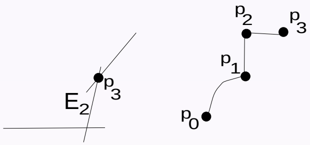
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.



We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

A divisor is determined by the combinatorics + moduli...

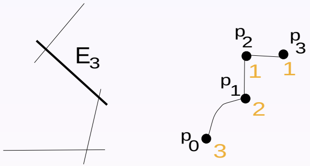
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.



We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

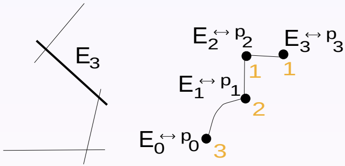
A divisor is determined by the combinatorics + moduli...

Minimal model of a divisor by blowing up a finite number of (free or satellite) points.



We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

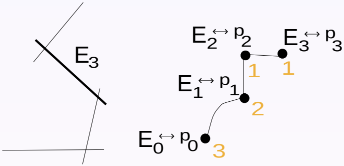
A divisor is determined by the combinatorics + moduli...
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.



We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

A divisor is determined by the combinatorics + moduli...

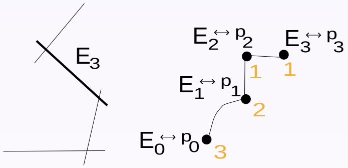
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.



These drawings only keep combinatorics, not moduli for free points.

We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

A divisor is determined by the combinatorics + moduli...
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.

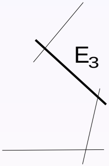


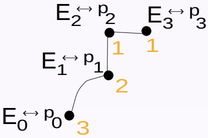
These drawings only keep combinatorics, not moduli for free points.

Embedded topology of plane curves is encoded in combinatorics of the minimal good embedded resolution.

We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

A divisor is determined by the combinatorics + moduli...
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.



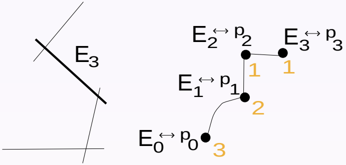


These drawings only keep combinatorics, not moduli for free points.

Embedded topology of plane curves is encoded in combinatorics of the minimal good embedded resolution.
Observe: not all the divisors are the final one for the minimal embedded resolution of a branch, but only the blow ups of satellite points.

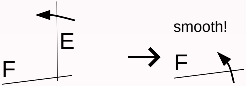
We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

A divisor is determined by the combinatorics + moduli...
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.



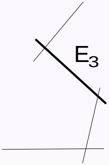
These drawings only keep combinatorics, not moduli for free points.

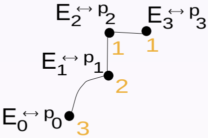
Embedded topology of plane curves is encoded in combinatorics of the minimal good embedded resolution.
Observe: not all the divisors are the final one for the minimal embedded resolution of a branch, but only the blow ups of satellite points.



We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

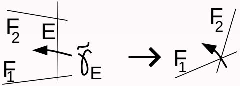
A divisor is determined by the combinatorics + moduli...
Minimal model of a divisor by blowing up a finite number of (free or satellite) points.





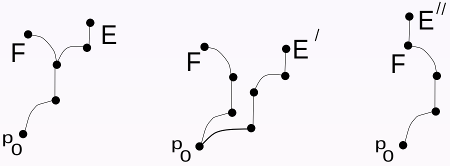
These drawings only keep combinatorics, not moduli for free points.

Embedded topology of plane curves is encoded in combinatorics of the minimal good embedded resolution.
Observe: not all the divisors are the final one for the minimal embedded resolution of a branch, but only the blow ups of satellite points.



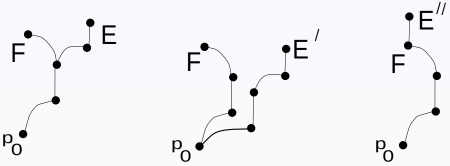
We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

Take into account the contact order between E and F, and much more (moduli for free points also count apriori!)...



We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

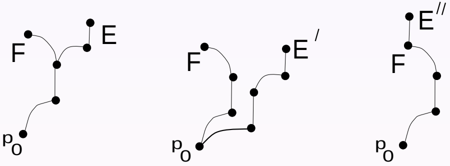
Take into account the contact order between E and F, and much more (moduli for free points also count apriori!)...



Talk about combinatorics of the pair (E, F). We write $(E, F) \equiv (E', F')$...

We want to describe $\overline{N}_E \subset \overline{N}_F$ (partial order among divisors over $O \in \mathbb{C}^2$)

Take into account the contact order between E and F, and much more (moduli for free points also count apriori!)...



Talk about combinatorics of the pair (E, F). We write $(E, F) \equiv (E', F')$...
Domination relation $F < E$ = Inclusion of Enriques diagrams.

Valuative criterion in arc spaces (A. Reguera, C. Plenat, S. Ishii...)

- Divisorial valuation ord_E = vanishing order along the divisor E.
- Can be computed intersecting with appropiate arcs γ in N_E:

$$ord_E(f) = I_O(f,\gamma) = ord_t(f \circ \gamma(t)).$$

- Choosing a family of arcs with an approriate $\alpha_0 \in N_E$ and $\alpha_s \in N_F$ we get

$$ord_F(h) \leq ord_t(h \circ \alpha_s(t)) \leq ord_t(h \circ \alpha_0(t)) = ord_E(h)$$

forall $h \in \mathbb{C}[[x,y]]$.

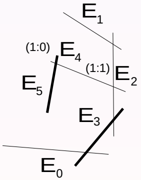
Valuative criterium in arc spaces (A. Reguera, C. Plènat, S. Ishii...)

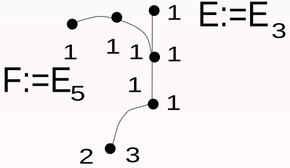
A. Reguera for rational surfaces in Manuscripta math. 1995.
C. Plénat in general in Annal Inst. Fourier 2005:

- $\overline{N}_E \subset \overline{N}_F$ implies $ord_E \leq ord_F$

S. Ishii in *Maximal Divisorial Sets*:

- if F is toric, also the converse is true: $ord_E \leq ord_F$ implies $\overline{N}_E \subset \overline{N}_F$.
- She found a counterexample for the converse in general ($ord_F \leq ord_E$ but $\overline{N}_E \not\subset \overline{N}_F$).





$$F < E \Rightarrow \overline{N}_E \subset \overline{N}_F \Rightarrow ord_F \leq ord_E$$

Other tools to rule out adjacencies?

- $\overline{N}_E \subsetneq \overline{N}_F$ implies $codim(\overline{N}_F) < codim(\overline{N}_E)$.

In [de Fernex, Ein, Ishii, Lazarsfeld, Mustata'200?]:

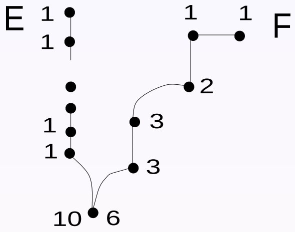
$$codim(\overline{N}_E) = 1 + disc(E, \mathbb{C}^2).$$

The *discrepancy* of E is the coefficient of E in $K_{X/\mathbb{C}^2}$ where $\pi : X \to \mathbb{C}^2$ is any model where E appears.

- It is not a sufficient criterium (even with toric examples)
- Neither plus the valuative criterium (counterexample of Ishii with the same discrepancy).

The problem turns very difficult...

Example of topological types.



- $ord_F \leq ord_E$
- $disc(E) + 1 = codim(\overline{N}_E) = 21 > codim(\overline{N}_F) = disc(F) + 1 = 17$
- $\overline{N}_E \not\subset \overline{N}_F$

Main result about inclusions

Recall that $\overline{N}_E \subset \overline{N}_F$ depends on the relative position of E and F, but ... we don't know a priori that it is a combinatorial problem, it also depends on the moduli of the free points!.

Theorem

Assume there exists a wedge α *realising the adjacency* $\overline{N}_E \subset \overline{N}_F$*. If* $(E', F') \equiv (E, F)$ *then there exists a wedge realising the adjacency* $\overline{N}_{E'} \subset \overline{N}_{F'}$*.*

Main result about inclusions

Recall that $\overline{N}_E \subset \overline{N}_F$ depends on the relative position of E and F, but ... we don't know a priori that it is a combinatorial problem, it also depends on the moduli of the free points!.

Theorem

Assume there exists a wedge α realising the adjacency $\overline{N}_E \subset \overline{N}_F$. If $(E', F') \equiv (E, F)$ then there exists a wedge realising the adjacency $\overline{N}_{E'} \subset \overline{N}_{F'}$.

Corollary

Assume we have that $\overline{N}_E \subset \overline{N}_F$. Let i_0 be the contact order between E and F. Then, we have that

$$\bigcup_{E' \equiv_{\geq i_0} E} \overline{N}_{E'} \subset \bigcap_{F' \equiv_{\geq i_0} F} \overline{N}_{F'}$$

where $A \equiv_{\geq i_0} B$ means that A has the same combinatorics as B and their contact order is $\geq i_0$.

Main result about inclusions

Recall that $\overline{N}_E \subset \overline{N}_F$ depends on the relative position of E and F, but ... we don't know a priori that it is a combinatorial problem, it also depends on the moduli of the free points!.

Theorem

Assume there exists a wedge α realising the adjacency $\overline{N}_E \subset \overline{N}_F$. If $(E', F') \equiv (E, F)$ then there exists a wedge realising the adjacency $\overline{N}_{E'} \subset \overline{N}_{F'}$.

Corollary

Assume we have that $\overline{N}_E \subset \overline{N}_F$. Let i_0 be the contact order between E and F. Then, we have that

$$\bigcup_{E' \equiv_{\geq i_0} E} \overline{N}_{E'} \subset \bigcap_{F' \equiv_{\geq i_0} F} \overline{N}_{F'}$$

where $A \equiv_{\geq i_0} B$ means that A has the same combinatorics as B and their contact order is $\geq i_0$.

We improve the log-discrepancy inequality in many cases.

Main result about inclusions

Recall that $\overline{N}_E \subset \overline{N}_F$ depends on the relative position of E and F, but ... we don't know a priori that it is a combinatorial problem, it also depends on the moduli of the free points!.

Theorem

Assume there exists a wedge α realising the adjacency $\overline{N}_E \subset \overline{N}_F$. If $(E', F') \equiv (E, F)$ then there exists a wedge realising the adjacency $\overline{N}_{E'} \subset \overline{N}_{F'}$.

Corollary

Assume we have that $\overline{N}_E \subset \overline{N}_F$. Let i_0 be the contact order between E and F. Then, we have that

$$\bigcup_{E' \equiv_{\geq i_0} E} \overline{N}_{E'} \subset \bigcap_{F' \equiv_{\geq i_0} F} \overline{N}_{F'}$$

where $A \equiv_{\geq i_0} B$ means that A has the same combinatorics as B and their contact order is $\geq i_0$.

We improve the log-discrepancy inequality in many cases.

Other conjectures...

$$(E,F) \equiv (E',F') \ \Rightarrow [\overline{N_E} \subsetneq \overline{N_F} \ \Leftrightarrow \ \overline{N_{E'}} \subsetneq \overline{N_{F'}}]$$

$(E,F) \equiv (E',F') \Rightarrow [\overline{N}_E \subsetneq \overline{N_F} \Leftrightarrow \overline{N}_{E'} \subsetneq \overline{N_{F'}}]$

To change the complex structure we can use:

- Let $g : X \to Y$ is a non-ramified covering of differentaible manifolds. A complex structure on Y can be lifted to X so that g is a local biholomorphism.

$(E, F) \equiv (E', F') \Rightarrow [\overline{N_E} \subsetneq \overline{N_F} \Leftrightarrow \overline{N_{E'}} \subsetneq \overline{N_{F'}}]$

To change the complex structure we can use:

- Let $g : X \to Y$ is a non-ramified covering of differentaible manifolds. A complex structure on Y can be lifted to X so that g is a local biholomorphism.
- (Grauert-Remmert) Let A be a normal analytic space, let $B \subset A$ be a closed analytic subset such that $A \setminus B$ is dense in A. Let
$$f : U \to A \setminus B$$
be a finite and étale analytic morphism. Then there exists a finite analytic extension
$$\bar{f} : V \to A$$
from a normal analytic space V. Moreover V is unique up to isomorphism.

$(E, F) \equiv (E', F') \Rightarrow [\overline{N_E} \subsetneq \overline{N_F} \Leftrightarrow \overline{N_{E'}} \subsetneq \overline{N_{F'}}]$

To change the complex structure we can use:

- Let $g : X \to Y$ is a non-ramified covering of differentaible manifolds. A complex structure on Y can be lifted to X so that g is a local biholomorphism.
- (Grauert-Remmert) Let A be a normal analytic space, let $B \subset A$ be a closed analytic subset such that $A \setminus B$ is dense in A. Let

  $$f : U \to A \setminus B$$

  be a finite and étale analytic morphism. Then there exists a finite analytic extension

  $$\bar{f} : V \to A$$

  from a normal analytic space V. Moreover V is unique up to isomorphism.

We can assume the wedge $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ is algebraic, that is there exists polynomials $F_1, F_2 \in \mathbb{C}[s, t, x, y]$ such that

$$F_1(s, t, \alpha_1(s, t)) = F_2(s, t, \alpha_2(s, t)) = 0.$$

Coming back to the valuative criterium...

Recall: $\overline{N}_E \subset \overline{N}_F$ implies there exists a family of parametrizations $\alpha(t,s)$ with $\alpha_0(t) \in \dot{N}_E$ and $\alpha_s \in \dot{N}_F$ for all $s \in \Lambda \setminus \{0\}$.

- Deforming a little α, we can assume that
$$\alpha^{-1}(O) = \{0\} \times \Lambda.$$
- The equation $F(x,y,s)$ of
$$Im[(t,s) \mapsto (\alpha(t,s),s) \in \mathbb{C}^2 \times \Lambda]$$
gives a deformation of plane curves given by $f_s(x,y) := F(x,y,s)$ where $f_0(x,y) = 0$ lifts transversally to E and all $f_s(x,y) = 0$ lift transversally to F for all $s \neq 0$.
- These deformations have a special property: for $s \neq 0$ they can be resolved simultaneously by a sequence of blow ups, **they fix the free points** (for F).

Valuative criterium.

- Let f_s be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some E and $f_s = 0$ have strict transforms transverse to a fixed F for all $s \neq 0$, then

  $$ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x, y]].$$

  We have $I_O(h, f_s) \leq I_O(h, f_0)$ but is not enough...

Valuative criterium.

- Let f_s be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some E and $f_s = 0$ have strict transforms transverse to a fixed F for all $s \neq 0$, then

$$ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x,y]].$$

  We have $I_O(h, f_s) \leq I_O(h, f_0)$ but is not enough...

Proof

Take embedded resolution $(\tilde{X}, D = \bigcup_i D_i) \to (\mathbb{C}^2, O)$ of $f_s = 0$ and $f_0 = 0$.
Look at it in family $\tilde{X} \times \Lambda \to \mathbb{C}^2 \times \Lambda$.
Let Y be the strict transform of $F = 0$ $(F(x,y,s) := f_s(x,y))$. Observe

$$Y_s = \{\widetilde{f_s = 0}\} \quad for \ s \neq 0$$

$$Y_0 = \{\widetilde{f_0 = 0}\} + \sum_k d_k D_k, \quad with \ \ d_k \geq 0.$$

We get $Y_0 \cdot D_i = Y_s \cdot D_i$ for any i. Putting $M = (D_i \cdot D_j)$, $(E = D_0,\ F = D_n)$,

$$(1, 0, ..., 0)^t + M(d_1, .., d_n)^t = (0, ..., 0, 1)^t.$$

$$-M^{-1}(1, 0, ..., 0, -1)^t = (d_1, ..., d_n)^t \geq 0.$$

and the entries of $-M^{-1}$ are exactly $ord_{D_i}(h_{D_j}) = I_O(h_{D_i}, h_{D_j})$.

Valuative criterium.

- Let f_s be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some E and $f_s = 0$ have strict transforms transverse to a fixed F for all $s \neq 0$, then

$$ord_F(h) \leq ord_E(h) \ \ \forall h \in \mathbb{C}[[x,y]].$$

  We have $I_O(h, f_s) \leq I_O(h, f_0)$ but is not enough...

- Reciprocally, if

$$ord_F(h) \leq ord_E(h) \ \ \forall h \in \mathbb{C}[[x,y]],$$

  then, taking $h_E = 0$ and $h_F = 0$ with strict transform transverse to E and F in a model of $E + F$, then

$$h_E + s \cdot h_F$$

  have strict transform transverse to F for $s \neq 0$ small enough. (Also proved by M. Alberich y J. Roe).

Valuative criterium.

- Let f_s be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some E and $f_s = 0$ have strict transforms transverse to a fixed F for all $s \neq 0$, then

$$ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x,y]].$$

  We have $I_O(h, f_s) \leq I_O(h, f_0)$ but is not enough...

- Reciprocally, if

$$ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x,y]],$$

  then, taking $h_E = 0$ and $h_F = 0$ with strict transform transverse to E and F in a model of $E + F$, then

$$h_E + s \cdot h_F$$

  have strict transform transverse to F for $s \neq 0$ small enough. (Also proved by M. Alberich y J. Roe).

Proof

Check it works.

Valuative criterium.

- Let f_s be a deformation fixing the free points. If $f_0 = 0$ has strict transform transverse to some E and $f_s = 0$ have strict transforms transverse to a fixed F for all $s \neq 0$, then
$$ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x, y]].$$
We have $I_O(h, f_s) \leq I_O(h, f_0)$ but is not enough...
- Reciprocally, if
$$ord_F(h) \leq ord_E(h) \quad \forall h \in \mathbb{C}[[x, y]],$$
then, taking $h_E = 0$ and $h_F = 0$ with strict transform transverse to E and F in a model of $E + F$, then
$$h_E + s \cdot h_F$$
have strict transform transverse to F for $s \neq 0$ small enough. (Also proved by M. Alberich y J. Roe).

Summarizing:

Proposition

Let E and F be two prime divisors. There exists a deformation f_s of a curve $f_0 = 0$ that lifts transversal to E that fixes the free points for F ($f_s = 0$ has strict transform transverse to F for $s \neq 0$) if and only if $ord_F \leq ord_E$.

So, the valuative criterion is a criterion for the existence of deformations of functions not of parametrizations!

Adjacency problems.

- CLASSICAL ONE: Given two topological types $f = 0$ and $g = 0$ in $(\mathbb{C}^2, 0)$, study when there exists a deformation $f_t = 0$ where $f_0 = 0$ has the topological type of $f = 0$ and $f_t = 0$ the one of $g = 0$.
- OUR OBSERVATION: deformation fixing the free points of the generic curves are characterized by the valuative criterium.

Valuative criterium.

Proposition

Let E and F be prime divisors over $O \in \mathbb{C}^2$.
There exists a deformation f_s of a curve $f_0 = 0$ that lifts transversal to E that fixes the free points for F ($f_s = 0$ has strict transform transverse to F for $s \neq 0$) if and only if $ord_F \leq ord_E$.

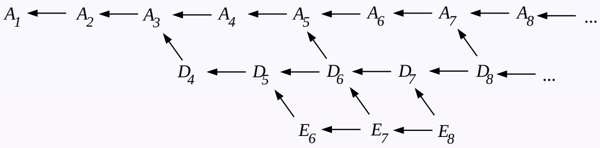
Good things about the result and our problem:

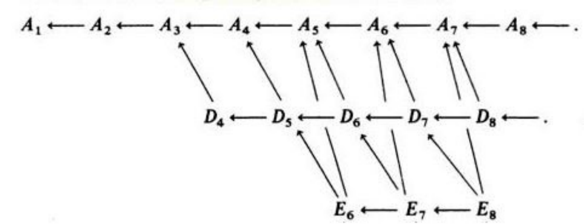
- It talks about concrete divisors, not only topological types.
- Takes into account the contact order of E and F.
- They are very easy to check finite conditions (inequalities for h_D with D in the minimal model of F) $=>$ Algorithm!
- Also works for F a non-prime divisor: if $F = \sum_i a_i F_i$ then we the condition is $ord_F := \sum_i a_i ord_{F_i} \leq ord_E$.

Bad news:

- Not all the adjacencies are of this type.

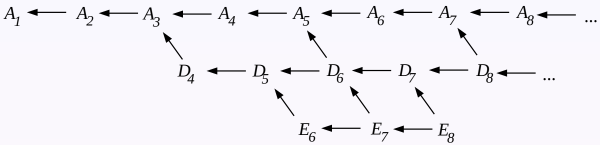
We recover many of the adjacencies from Arnol'd's list.





Only 7 out of the 93 classical adjacencies between simple singularities of $\mu \leq 8$ are not realizable.

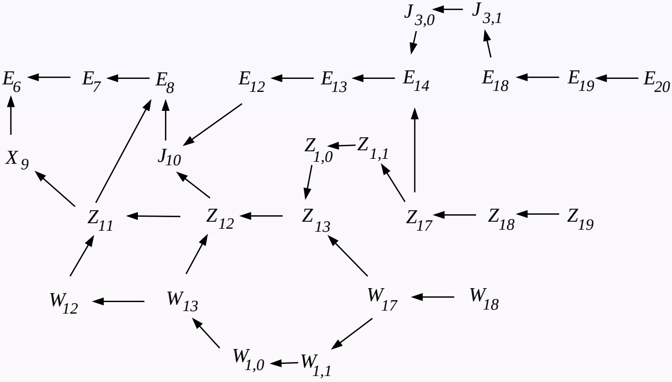
We recover many of the adjacencies from Arnol'd's list.



- For example, $ord_{A_5} \not\leq ord_{E_6}$ but still there exists a deformation

$$y^3 + x^4 + s^2y^2 + 2sx^2y.$$

We recover many of the adjacencies from Arnol'd's list.



Some were not in Arnol'd's list:

$$Z_{11} = S_{2,4,5} \to E_8,\ Z_{12} = S_{2,4,6} \to J_{10} = T_{2,3,6},\ W_{17} \to Z_{13} = S_{2,4,7}$$

Some are not realizable:

$$W_{18} \not\to Z_{17},\ Z_{11} \not\to J_{10},\ X_9 \not\to E_7.$$

Relation to the study of δ constant stratum.

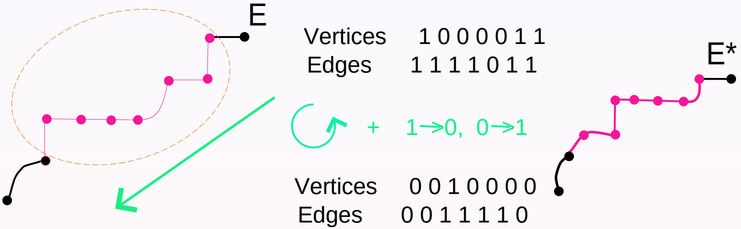
- Recall Teissier's Theorem: a deformation f_t admits a parametrization in family if and only if it is δ-constant. $(\delta(C,0) = dim_{\mathbb{C}}(\mathcal{O}_{\bar{C},\bar{0}}/\mathcal{O}_{C,0}))$.
- Describe all the $\overline{N}_E \subset \overline{N}_F$ is equivalent to describe which of the deformations fixing the free points are in the δ-constant stratum.
- Our problem is slightly different to the classical study of the δ-constant stratum: may be easier?

Happy birthday and thank you!

Order and duality for topological types that are resolved in n blow-ups.

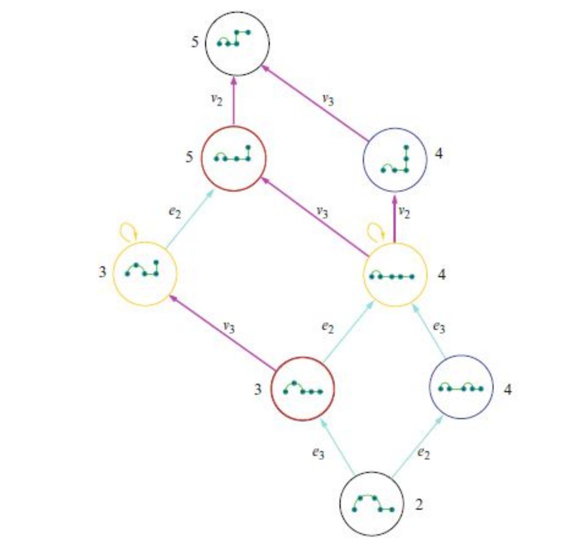
Order and duality for topological types that are resolved in n blow-ups.

Take combinatorial information (1/0) about $n-3$ edges (straight/curve) and $n-3$ vertices (broken between straight/smooth).
Combinatorics induces a partial order: the more straight lines and broken vertices, the bigger.

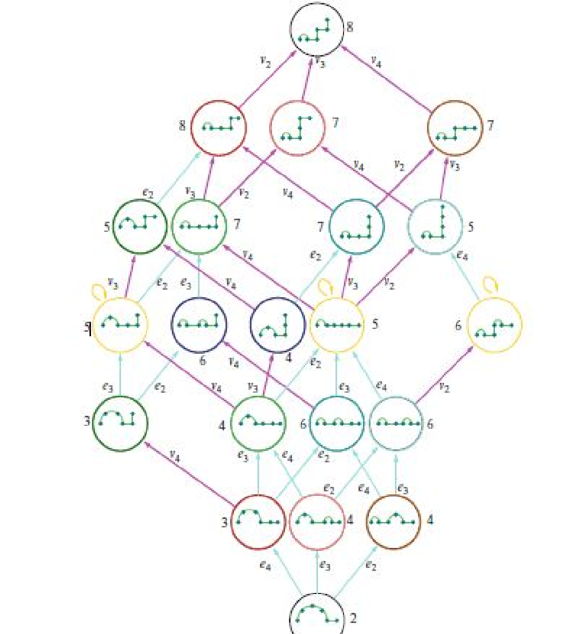


You get a duality that inverting the partial order just interchanging broken/curve and smooth/straight and reading backwards.

Order and duality for topological types that are resolved in n blow-ups.



Order and duality for topological types that are resolved in n blow-ups.



It is just a combinatorial happening for the moment, will it appear in a deeper context?

P. Popescu-Pampu, M. Pe Pereira, *Fibonacci numbers and self-dual lattice structures for plane branches.*Bridging Algebra, Geometry, and Topology , Springer Proceedings in Mathematics Statistics, 96