Enumeration of curves via non-archimedean geometry

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Happy birthday to Bernard Teissier !

Enumeration of curves via non-archimedean geometry

1. What is enumerative geometry about?
2. New results on tropical geometry and non-archimedean geometry
3. Count holomorphic cylinders in log Calabi-Yau surfaces
   - New geometric invariants
   - The wall-crossing formula conjectured by Kontsevich-Soibelman
4. Potential applications to singularity theory

Counting curves

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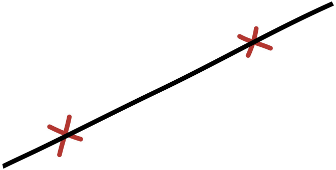

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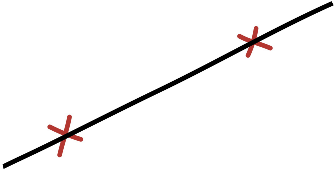

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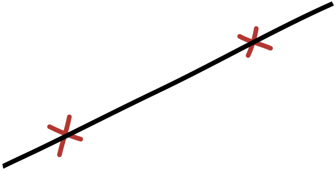

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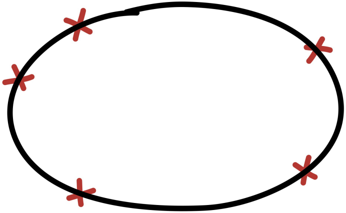

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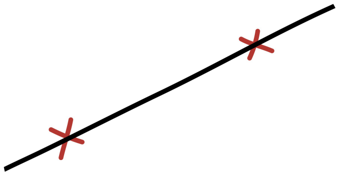

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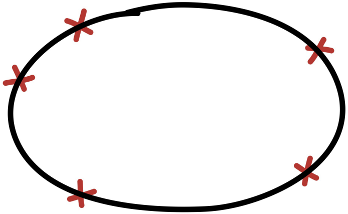

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Theorem (Kontsevich-Manin 94, Ruan-Tian 94)

The numbers N_d satisfy the following recursive formula

$$N_d = \sum_{\substack{d_1, d_2 > 0 \\ d_1 + d_2 = d}} \binom{3d-4}{3d_1-2} (d_1 d_2)^2 N_{d_1} N_{d_2} - \sum_{\substack{d_1, d_2 > 0 \\ d_1 + d_2 = d}} \binom{3d-4}{3d_2-1} d_1 d_2^3 N_{d_1} N_{d_2}.$$

Now we can compute $N_5 = 87304$, $N_6 = 26312976$, $N_7 = 14616808192$, $N_8 = 13525751027392$, $N_9 = 19385778269260800, \dots$

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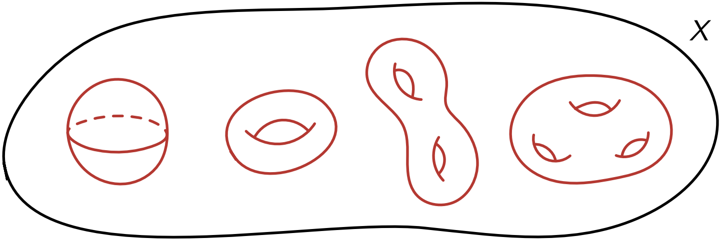
Counting rational curves in $\mathbb{CP}^2$ is an example of enumerative geometry. The main themes of enumerative geometry are

- Define new enumerative invariants.
- Study their properties.

Besides rational curves in $\mathbb{CP}^2$, we can also count curves with higher genus in a general smooth projective variety X :



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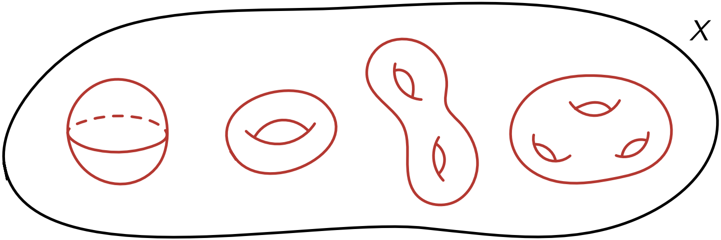
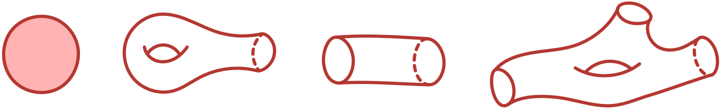


Moreover, theoretical physics (in particular, mirror symmetry) suggests that besides counting "closed curves", it is also important to count "open curves" (i.e. Riemann surfaces with boundaries):



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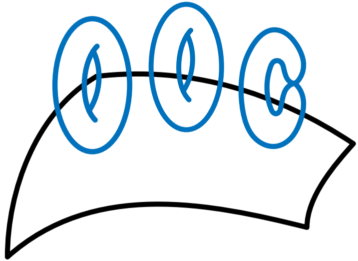

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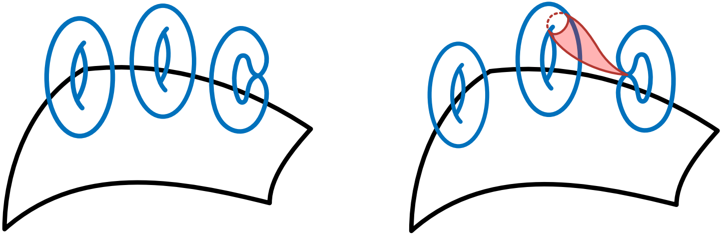
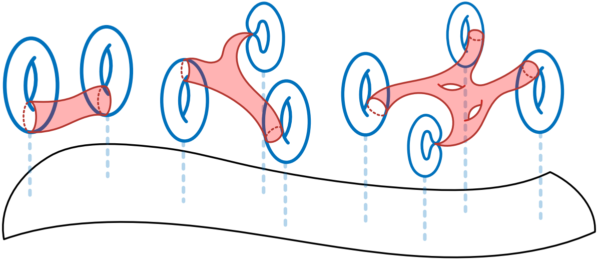


It is of great interest to count holomorphic discs in the total space, with boundaries on a torus fiber,

and more generally, Riemann surfaces with boundaries on the torus fibers.



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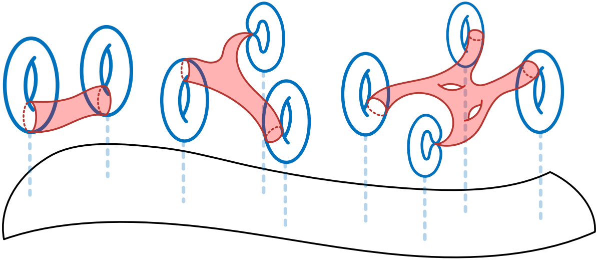

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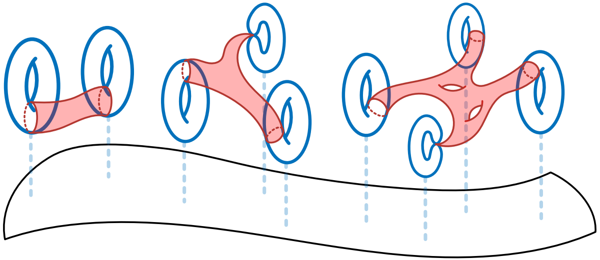

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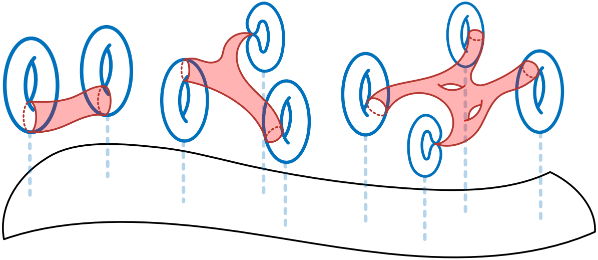

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- The existence of the SYZ torus fibration is largely conjectural (Gross-Wilson 2000).
- The counting of holomorphic discs does not give simple numerical invariants. (It does give rise to more sophisticated structures, e.g. obstructions in Floer homology (Fukaya-Oh-Ohta-Ono).)

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Theorem (Berkovich 99)

Given a nice formal model of X, one can construct a strong deformation retraction from X to a polyhedral complex S embedded in X, called the skeleton.

(see also the works by Thuillier, Mustata-Nicaise, Nicaise-Xu)

Example of K3 surface

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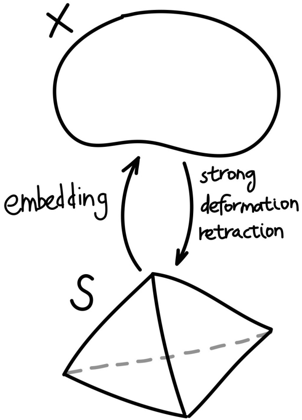
X : K3 surface of type III degeneration

The skeleton S is a polyhedral complex homeomorphic to S^2.



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A: I still do not know how to do it in general. But I developed some preliminary steps and studied a particular case for log Calabi-Yau surfaces.

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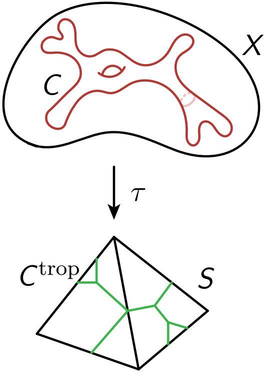

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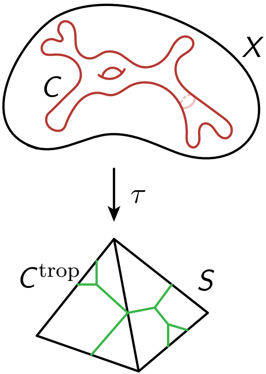

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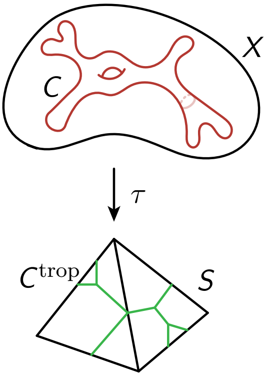

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The balancing conditions are constraints on the shape of C^{trop} around every vertex.

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A: To be more precise, fix a real number A, and consider

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The proofs use formal models, Artin's representability theorem, the geometry of stable curves, de Jong's three point lemma, analytic étale cohomology, vanishing cycles and quantifier eliminations for rigid subanalytic sets.

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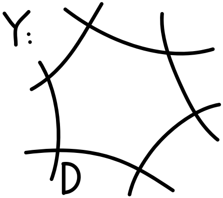

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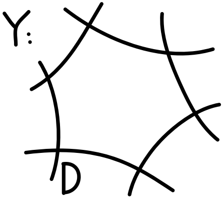

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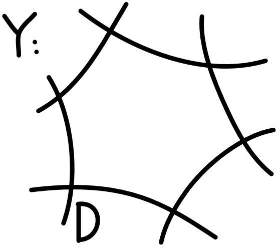
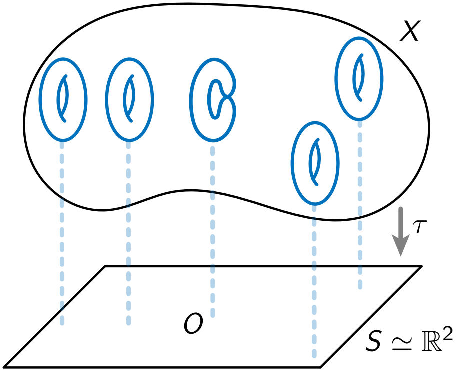
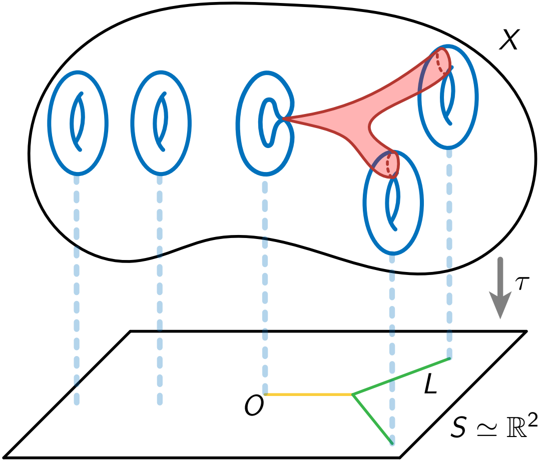




Goal: Define and study the enumeration of cylinders with boundaries on two fiber tori:



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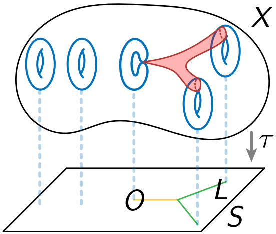

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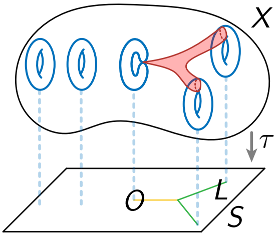

The last step turns out very difficult in non-archimedean geometry. But in the case of log Calabi-Yau surfaces, we can borrow the algebraic construction of virtual fundamental class (Behrend-Fantechi), as long as we can prove the GAGA theorem for non-archimedean analytic stacks.

Theorem (Porta-Y 2014)

Let X be a (higher) algebraic stack proper over a k-affinoid space. Then the analytification functor induces an equivalence of categories

$$\mathrm{Coh}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{Coh}^{\mathrm{b}}(X^{\mathrm{an}}),$$

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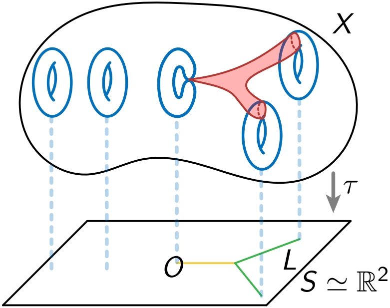

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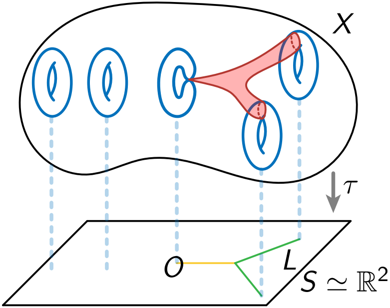
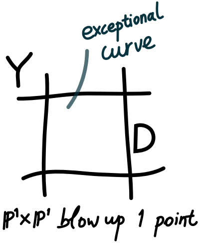


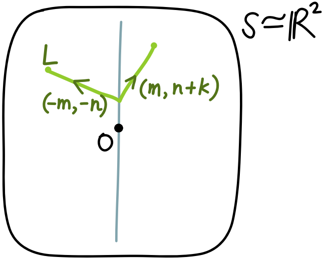
The number of cylinders enjoys very nice properties. I hope to explain to you in the future.

Now let's look at a concrete example for a del Pezzo surface.

Example (del Pezzo surface of degree 7)

Y : $\mathbb{CP}^1 \times \mathbb{CP}^1$ blowup a smooth point in the toric boundary:





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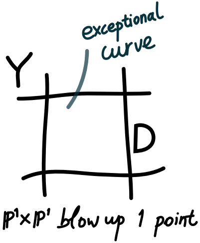

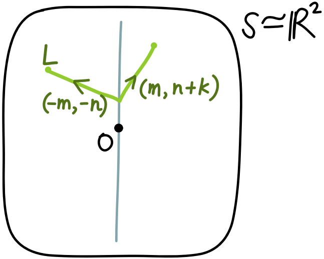

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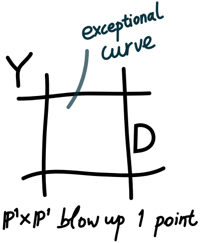

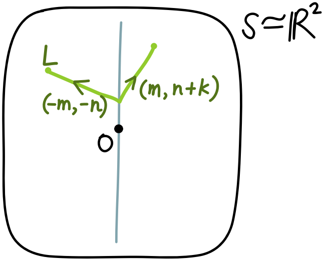

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It gives exactly the wall-crossing formula around a focus-focus singularity, conjectured by Kontsevich-Soibelman:

$$(x, y) \longmapsto (x(1 + y), y), \quad x^m y^n \longmapsto x^m(1 + y)^m y^n = \sum_{k=0}^{m} \binom{m}{k} x^m y^{k+n}.$$

Potential applications to singularity theory

Work in progress:

The number of cylinders $\longrightarrow$ a universal family of log CY surfaces.

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2. In the recent work *Mirror symmetry for log Calabi-Yau surfaces*, Gross, Hacking and Keel constructed the universal family mentioned above and proved Looijenga's conjecture on the smoothing of 2-dimensional cusp singularities.
   They do not use non-archimedean geometry. Their main tools are scattering diagrams and broken lines; both are combinatorial notions.
   Our counting of cylinders gives geometric interpretations to their combinatorial constructions.

3. In the near future, we aim to develop the enumeration of cylinders in log CY varieties of higher dimensions. We hope that this will provide a new tool for the study of moduli spaces and singularities in higher dimensions.

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Thank you very much for your attention!