Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs

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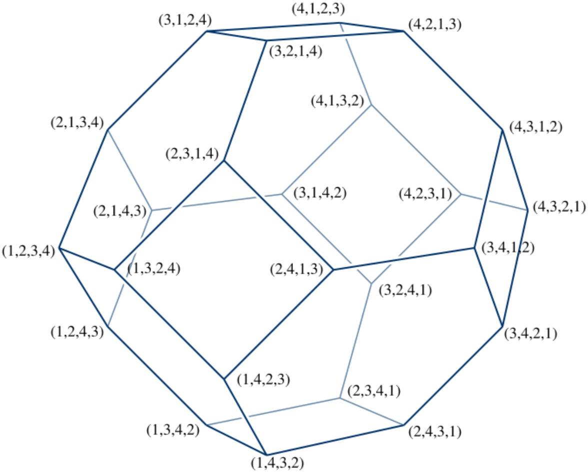
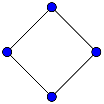
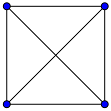
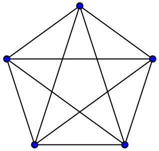
The 3-dimensional permutohedron P_3:

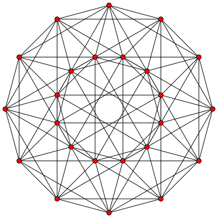
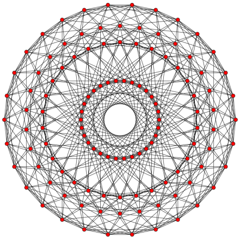


This polytope has the symmetry of the root system A_3.

A *graph* is a 1-dimensional space, with vertices and edges.

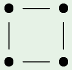
Graphs are the simplest geometric structures.

Whitney (1932): The *chromatic polynomial* of a graph G is the function

$$\chi_G(q) = (\text{the number of proper colorings of } G \text{ with } q \text{ colors}).$$

Example



$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q, \qquad \chi_G(2) = 2.$$

What can be said about the chromatic polynomial in general?

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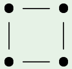


$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q, \qquad \chi_G(2) = 2.$$

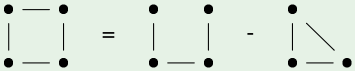
Read's conjecture (1968)

The absolute values of the coefficients of the chromatic polynomial $\chi_G(q)$ form a log-concave sequence for any graph G, that is,

$$a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } i.$$

Example

How do we compute the chromatic polynomial? We write



and use

$$\begin{aligned} \chi_{G\setminus e}(q) &= q(q-1)^3 \\ \chi_{G/e}(q) &= q(q-1)(q-2). \end{aligned}$$

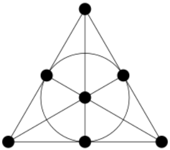
Therefore

$$\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q) = 1q^4 - 4q^3 + 6q^2 - 3q.$$

This algorithmic description of $\chi_G(q)$ makes the prediction of the conjecture interesting.

For any finite set of vectors A in a vector space over a field, define

$$f_i(A) = (\text{number of independent subsets of } \mathscr{A} \text{ with size } i).$$



Example

If A is the set of all nonzero vectors in $\mathbb{F}_2^3$, then

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

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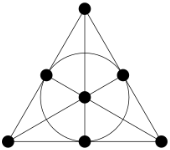

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How do we compute $f_i(A)$? We use

$$f_i(A) = f_i(A \setminus v) + f_{i-1}(A / v).$$

Welsh's conjecture (1969)

The sequence f_i form a log-concave sequence for any finite set of vectors A in any vector space over any field, that is,

$$f_i^2 \geq f_{i-1}\, f_{i+1} \text{ for all } i.$$

Whitney (1935).

A *matroid* on a finite set E is a collection of subsets of E, called *independent sets*, which satisfy axioms modeled on the relation of linear independence of vectors:

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A *matroid* on a finite set E is a collection of subsets of E, called *independent sets*, which satisfy axioms modeled on the relation of linear independence of vectors:

1. Every subset of an independent set is an independent set.
2. If an independent set A has more elements than independent set B, then there is an element in A which, when added to B, gives a larger independent set.

1. Let V be a vector space over a field k, and A a finite set of vectors.

   Call a subset of A independent if it is linearly independent.

   This defines a matroid M *realizable* over k.

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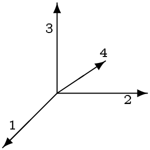
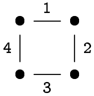
2. Let G be a finite graph, and E the set of edges.

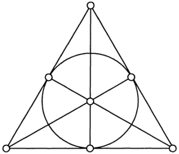
   Call a subset of E independent if it does not contain a circuit.

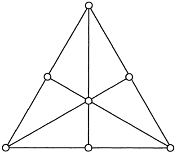
   This defines a *graphic matroid* M.



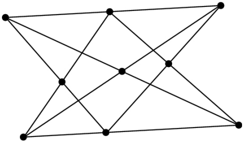


Fano matroid is realizable iff char$(k) = 2$.





Non-Fano matroid is realizable iff char$(k) \neq 2$.



Non-Pappus matroid is not realizable over any field.

Testing the realizability of a matroid is not easy: When $k = \mathbb{Q}$, this is equivalent to Hilbert's tenth problem over $\mathbb{Q}$.

One can define the *chromatic polynomial* of a matroid by the recursion

$$\chi_M(q) = \chi_{M\setminus e}(q) - \chi_{M/e}(q).$$

Rota's conjecture (1970)

The coefficients of the chromatic polynomial $\chi_M(q)$ form a log-concave sequence for any matroid M, that is,

$$\mu_i^2 \geq \mu_{i-1}\mu_{i+1} \text{ for all } i.$$

This implies the conjecture on G and the conjecture on A.

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How to show that a sequence is log-concave?

- h : a nonconstant homogeneous polynomial in $\mathbb{C}[z_0, \ldots, z_r]$.
- J_h : the jacobian ideal $\big(\partial h/\partial z_0, \ldots, \partial h/\partial z_n\big)$.
- Define the numbers $\mu^i(h)$ by saying that the function

$$\dim_{\mathbb{C}} \mathfrak{m}^u J_h^v / \mathfrak{m}^{u+1} J_h^v$$

agrees with the polynomial for large enough u and v

$$\frac{\mu^0(h)}{r!} u^r + \cdots + \frac{\mu^i(h)}{(r-i)!i!} u^{r-i} v^i + \cdots + \frac{\mu^r(h)}{r!} v^r + \text{(lower degree terms).}$$

Theorem (-, 2012)

For any nonconstant homogeneous polynomial $h \in \mathbb{C}[z_0, \ldots, z_r]$,

1. $\mu^i(h)$ *is the number of* i*-dimensional cells in a CW-model of the complement*

$$D(h) := \{x \in \mathbb{P}^r \mid h(x) \neq 0\}.$$

2. $\mu^i(h)$ *form a log-concave sequence, and*
3. *if* h *is product of linear forms, then the attaching maps are homologically trivial:*

$$\mu^i(h) = b_i\big(D(h)\big).$$

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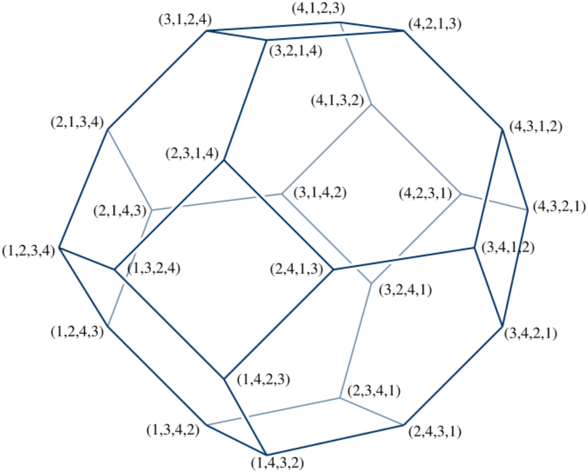
When h defines a hyperplane arrangement $\mathscr{A}$, this gives

$$\mu^i(h) = \mu_i(\mathscr{A}) := (\text{the } i\text{-th coefficient of the characteristic polynomial of } \mathscr{A}),$$

justifying the log-concavity for matroids realizable over a field of characteristic zero.

Matroids on $[n] = \{0, 1, \ldots, n\}$ are closely related to the geometry of the toric variety X_{A_n} of the n-dimensional permutohedron:



- The rays of its normal fan Δ_{A_n} correspond to nonempty proper subsets of $[n]$.
- More generally, k-dimensional cones of Δ_{A_n} correspond to flags of nonempty proper subsets of $[n]$:

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k.$$

- The "extra symmetry" of P_n maps a flag

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k.$$

to the flag of complements

$$[n] \setminus S_1 \supsetneq [n] \setminus S_2 \supsetneq \cdots \supsetneq [n] \setminus S_k.$$

- A matroid M of rank $r+1$ on $[n]$ can be viewed as an r-dimensional subfan

$$\Delta_M \subseteq \Delta_{A_n}$$

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which consists of cones corresponding to flags of flats of M:

$$F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r.$$

- The fan Δ_M is the *Bergman fan* of M,

or the *tropical linear space* associated to M.

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What we show is that the tropical variety Δ_M has a "cohomology ring" which has the structure of the cohomology ring of a smooth projective variety.

(I would guess that most of these "cohomology rings" of matroids are not isomorphic to the cohomology ring of any smooth projective variety, but I do not know this.)

A motivating observation is that the toric variety of Δ_M is, in the realizable case, 'Chow equivalent' to a smooth projective variety:

There is a map from a smooth projective variety

$$V \longrightarrow X_{\Delta_M}$$

which induces an isomorphism between Chow cohomology rings

$$A^*(X_{\Delta_M}) \longrightarrow A^*(V).$$

It is tempting to think this as a 'Chow homotopy'.

In fact, the converse also holds.

Theorem

The toric variety X_{Δ_M} is Chow equivalent to a smooth projective variety over k if and only if M is realizable over the field k.

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The toric variety X_{Δ_M} is Chow equivalent to a smooth projective variety over k if and only if M is realizable over the field k.

We show that, even in the non-realizable case, $A^*(M) := A^*(X_{\Delta_M})$ has the structure of the cohomology ring of a smooth projective variety.

The proof is a good advertisement for tropical geometry to pure combinatorialists:

For any two matroids on $[n]$ with the same rank, there is a diagram

$$\Delta_M \xrightarrow{\text{“\textit{flip}”}} \Delta_1 \xrightarrow{\text{“\textit{flip}”}} \Delta_2 \xrightarrow{\text{“\textit{flip}”}} \cdots \xrightarrow{\text{“\textit{flip}”}} \Delta_{M'} \ ,$$

and each flip preserves the validity of the Kähler package in the cohomology ring.

The intermediate objects are tropical varieties with good cohomology rings,

but not in general associated to a matroid.

The cohomology ring $A^*(M)$ can be described explicitly by generators and relations, which can be taken as a definition.

Definition

The cohomology ring of M is the quotient of the polynomial ring

$$A^*(M) := \mathbb{Z}[x_F]/(I_1 + I_2),$$

where the variables are indexed by nonempty proper flats of M, and

$$I_1 \quad := \quad \text{ideal}\left(\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F \mid i_1 \text{ and } i_2 \text{ are distinct elements of } [n] \right),$$

$$I_2 \quad := \quad \text{ideal}\left(x_{F_1} x_{F_2} \mid F_1 \text{ and } F_2 \text{ are incomparable flats of } M \right).$$

Proposition

The Chow ring $A^*(M)$ is a Poincaré duality algebra of dimension r:

(1) *Degree map: There is an isomorphism*

$$\deg : A^r(M) \longrightarrow \mathbb{Z}, \qquad \prod_{i=1}^{r} x_{F_i} \longmapsto 1,$$

for any complete flag of nonempty proper flats $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r$ of M.

(2) *Poincaré duality: For any nonnegative integer $k \leq r$, the multiplication defines the perfect pairing*

$$A^k(M) \times A^{r-k}(M) \longrightarrow A^r(M) \simeq \mathbb{Z},$$

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Note that the underlying simplicial complex of Δ_M, the *order complex* of M, is not Gorenstein in general.

Digression: Why can't we prove (at the moment) the g-conjecture for simplicial spheres?

Because we do not understand Kähler classes in their cohomology ring.

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The case of non-realizable matroids contrasts this in an interesting way.

Let $\mathscr{K}_{[n]}$ be the convex cone of linear forms with real coefficients

$$\mathscr{K}_{[n]} := \left\{ \sum_S c_S x_S \mid S \text{ is a nonempty proper subset of } [n] \right\}$$

consisting of linear forms satisfying

$$c_{S_1} + c_{S_2} > c_{S_1 \cap S_2} + c_{S_1 \cup S_2} \qquad (c_\emptyset = c_{[n]} = 0),$$

for any two incomparable nonempty proper subsets S_1, S_2 of $[n]$.

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Definition

The *ample cone* of M, denoted $\mathscr{K}_M$, is defined to be the image

$$\mathscr{K}_{[n]} \longrightarrow \mathscr{K}_M \subseteq A^1(M)_\mathbb{R},$$

where the non-flats of M are mapped to zero.

Theorem (AHK)

Let ℓ be an element of $\mathscr{K}_M$ and let k be a nonnegative integer $\leq r/2$.

(1) *Hard Lefschetz: The multiplication by ℓ defines an isormophism*

$$A^k(M)_{\mathbb{R}} \longrightarrow A^{r-k}(M)_{\mathbb{R}}, \qquad h \longmapsto \ell^{r-2k} \cdot h.$$

(2) *Hodge-Riemann: The multiplication by ℓ defines a definite form of sign $(-1)^k$*

$$PA^k(M)_{\mathbb{R}} \times PA^k(M)_{\mathbb{R}} \longrightarrow A^r(M)_{\mathbb{R}} \simeq \mathbb{R}, \qquad (h_1, h_2) \longmapsto \ell^{r-2k} \cdot h_1 \cdot h_2,$$

where $PA^k(M)_{\mathbb{R}} \subseteq A^k(M)_{\mathbb{R}}$ is the kernel of the multiplication by ℓ^{r-2k+1}.

Why does this imply the log-concavity conjecture?

Let i be an element of $[n]$, and consider the linear forms

$$\alpha(i) := \sum_{i \in S} x_S,$$
$$\beta(i) := \sum_{i \notin S} x_S.$$

Note that these linear forms are ‘almost’ ample:

$$c_{S_1} + c_{S_2} \geq c_{S_1 \cap S_2} + c_{S_1 \cup S_2} \qquad (c_\emptyset = c_{[n]} = 0).$$

Their images in the cohomology ring $A^*(M)$ does not depend on i, and

will be denoted by α and β respectively.

Proposition

Under the isomorphism deg : $A^r(M) \longrightarrow \mathbb{Z}$, *we have*

$$\alpha^{r-k}\beta^k \longmapsto (k\text{-th coefficient of the reduced characteristic polynomial of } M).$$

While neither α nor β are in the ample cone $\mathscr{K}_M$, we may take the limit

$$\ell_1 \longrightarrow \alpha, \qquad \ell_2 \longrightarrow \beta, \qquad \ell_1, \ell_2 \in \mathscr{K}_M.$$

This may be one reason why direct combinatorial reasoning for log-concavity was not easy.

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Corollary

The coefficients of the chromatic polynomial $\chi_M(q)$ *form a log-concave sequence for any matroid* M.