Lipschitz stratification in power-bounded o-minimal fields

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• Stratification

Let $X \subseteq \mathbb{R}^n$ be a subset. A **stratification** of X is a family

$$\mathcal{X} = (X^0 \subseteq X^1 \subseteq \cdots \subseteq X^d = X)$$

of subsets of X such that

- $\dim X^i \leq i$ for $0 \leq i \leq d$,
- $\mathring{X}^i := X^i \setminus X^{i-1}$, called the ***i*-th skeleton**, is either empty or a differentiable submanifold of $\mathbb{R}^n$ of dimension i (not necessarily connected), and each connected component of $\mathring{X}^i$ is called a **stratum**,
- For each stratum S, $\operatorname{cl} S \subseteq S \cup X^{i-1}$ is a union of strata.

• Projections to tangent spaces

For each point $a \in \mathring{X}^i$, let

$$P_a : \mathbb{R}^n \longrightarrow \boldsymbol{T}_a\mathring{X}^i \quad \text{and} \quad P_a^{\perp} \coloneqq \mathrm{id} - P_a : \mathbb{R}^n \longrightarrow \boldsymbol{T}_a^{\perp}\mathring{X}^i$$

be the orthogonal projections onto the tangent and the normal spaces of $\mathring{X}^i$ at a.

- **Verdier's condition**

Let $\mathcal{X} = (X^i)$ be a stratification of X. For every i and every $a \in \mathring{X}^i$ there are

- an (open) neighborhood $U_a \subseteq X$ of a,
- a constant C_a

such that, for

- every $j \geq i$,
- every $b \in \mathring{X}^i \cap U_a$,
- every $c \in \mathring{X}^j \cap U_a$

we have

$$\|P_c^{\perp} P_b\| \leq C_a \|c - b\|.$$

• In terms of vector fields

Let $\mathcal{X} = (X^i)$ be a stratification of X. A vector field v on an open subset $U \subseteq X$ is **$\mathcal{X}$-rugose** if

- v is tangent to the strata of $\mathcal{X}$ (**$\mathcal{X}$-compatible** for short),
- v is differentiable on each stratum of $\mathcal{X}$,
- for every $a \in \mathring{X}^i \cap U$ there is a constant C_a such that, for every $j \geq i$, all $b \in \mathring{X}^i \cap U$ and $c \in \mathring{X}^j \cap U$ that are sufficiently close to a satisfy
$$\|v(b) - v(c)\| \leq C_a \|b - c\|.$$

• Concerning Verdier's condition

THEOREM. • *(Verdier) Every subanalytic set admits a stratification that satisfies Verdier's condition.*

• *(Loi) The above holds in all o-minimal structures.*

THEOREM (Brodersen–Trotman). *$\mathcal{X}$ is Verdier if and only if each rugose vector field on $U \cap X^i$ can be extended to a rugose vector field on a neighborhood of $U \cap X^i$ in X.*

In general Verdier's condition is strictly stronger than Whitney's condition (b). But we do have:

THEOREM (Teissier). *For complex analytic stratifications, Verdier's condition is equivalent to Whitney's condition (b).*

• Concerning Mostowski's condition

Mostowski's condition is a (much) stronger condition than Verdier's condition.

THEOREM (Parusinski). *$\mathcal{X}$ is Lipschitz if and only if there is a constant C such that, for every $X^{i-1} \subseteq W \subseteq X^i$, if v is an $\mathcal{X}$-compatible Lipschitz vector field on W with constant L and is bounded on the last stratum of $\mathcal{X}$ by a constant K, then v can be extended to a Lipschitz vector field on X with constant $C(K+L)$.*

THEOREM (Parusinski). *Lipschitz stratifications exist for compact subanalytic subsets in $\mathbb{R}$.*

Main ingredients of the proof: local flattening theorem, Weierstrass preparation for subanalytic functions, and more.

Theorem (Nguyen–Valette). *Lipschitz stratifications exist for all definable compact sets in all polynomial-bounded o-minimal structures on the real field* $\mathbb{R}$.

Their proof follows closely and improves upon Parusinski's proof strategy; in particular, it refines a version of the Weierstrass preparation for subanalytic functions (van den Dries– Speissegger).

On the other hand, our result states:

Theorem. *Lipschitz stratifications exist for all definable closed sets in all power-bounded o-minimal structures (for instance, in the Hahn field* $\mathbb{R}(\!(t^{\mathbb{Q}})\!)$*).*

Our proof bypasses all of the machineries mentioned above and goes through analysis of definable sets in non-archimedean *o*-minimal structures instead.

• o-minimality

Definition. Let $\mathcal{L}$ be a language that contains a binary relation $<$. An $\mathcal{L}$-structure M is said to be ***o*-minimal** if

- $<$ is a total ordering on M,
- every definable subset of the affine line is a *finite* union of intervals (including points).

An $\mathcal{L}$-theory T is o-minimal if every one of its models is o-minimal.

• Two fundamental *o*-minimal structures

THEOREM (Tarski). *The theory RCF of the real closed field (essentially the theory of semialgebraic sets)*

$$\bar{\mathbb{R}} = (\mathbb{R}, <, +, \times, 0, 1)$$

is o-minimal.

THEOREM (Wilkie). *The theory $RCF_{\exp}$ of the real closed field with the exponential function*

$$\mathbb{R}_{\exp} = (\mathbb{R}, <, +, \times, 0, 1, \exp)$$

is o-minimal.

• Polynomial / power bounded structures

Let $\mathcal{R}$ be an o-minimal structure that expands a real closed field.

DEFINITION. A **power function** in $\mathcal{R}$ is a definable endomorphism of the multiplicative group of $\mathcal{R}$. (Note that such a power function f is uniquely determined by its **exponent** $f'(1)$.)

We say that $\mathcal{R}$ is **power-bounded** if every definable function in one variable is eventually dominated by a power function.

THEOREM (Miller). *Either M is power bounded or there is a definable exponential function in M (meaning a homomorphism from the additive group to the multiplicative group).*

Note: In $\mathbb{R}$, power-bounded becomes polynomial-bounded.

• Examples of polynomial-bounded o-minimal structures on $\mathbb{R}$

- RCF. (Semialgebraic sets).
- RCF_{an}: The theory of real closed fields with restricted analytic functions $f|_{[-1,1]^n}$. (Subanalytic sets).
- $RCF_{an,powers}$: RCF_{an} plus all the powers (x^r for each $r \in \mathbb{R}$).
- Further expansions of RCF_{an} by certain quasi-analytic functions
  - certain Denjoy-Carleman classes,
  - Gevrey summable functions,
  - certain solutions of systems of differential equations.

• Mostowski's condition (quantitative version)

Fix a (complete) *o*-minimal theory T (not necessarily power bounded). Let $\mathcal{R}$ be a model of T, for example,

$$\mathbb{R}, \quad \mathbb{R}((t^{\mathbb{Q}})), \quad \mathbb{R}((t_1^{\mathbb{Q}}))((t_2^{\mathbb{R}})), \quad \text{etc.}$$

The Mostowski condition is imposed on certain finite sequences of points called chains. The notion of a chain depends on several constants, which have to satisfy further conditions on additional constants.

In $\mathcal{R}$, let X be a definable set and $\mathcal{X} = (X^i)$ a definable stratification of X.

Definition. Let $c, c', C', C'' \in \mathcal{R}$ be given. A **(c, c', C', C'')-chain** is a sequence of points $a^0, a^1, \ldots, a^m$ in X with

$$a^\ell \in \mathring{X}^{e_\ell} \quad \text{and} \quad e_0 > e_1 > \cdots > e_m$$

such that the following holds.

- For $\ell = 1, \ldots, m$, we have:

$$\|a^0 - a^\ell\| < c \cdot \mathrm{dist}(a, X^{e_\ell})$$

- For each i with $e_m \leq i \leq e_0$, (exactly) one of the two following conditions holds:

$$\begin{cases} \mathrm{dist}(a^0, X^{i-1}) \geq C' \cdot \mathrm{dist}(a^0, X^i) & \text{if } i \in \{e_0, \ldots, e_m\} \\ \mathrm{dist}(a^0, X^{i-1}) < c' \cdot \mathrm{dist}(a^0, X^i) & \text{if } i \notin \{e_0, \ldots, e_m\}. \end{cases}$$

An **augmented (c, c', C', C'')-chain** is a (c, c', C', C'')-chain together with an additional point $a^{00} \in \mathring{X}^{e_0}$ satisfying

$$C''\|a^0 - a^{00}\| \leq \operatorname{dist}(a^0, X^{e_0-1}).$$

Definition. We say that the stratification $\mathcal{X} = (X^i)$ **satisfies the Mostowski condition for the quintuple** (c, c', C', C'', C''') if the following holds.

For every (c, c', C', C'')-chain (a^i),

$$\|P_{a^0}^{\perp} P_{a^1} \dots P_{a^m}\| < \frac{C'''\|a^0 - a^1\|}{\mathrm{dist}(a^0, X^{e_m - 1})}.$$

For every augmented (c, c', C', C'')-chain $((a^i), a^{00})$,

$$\|(P_{a^0} - P_{a^{00}}) P_{a^1} \dots P_{a^m}\| < \frac{C'''\|a^0 - a^{00}\|}{\mathrm{dist}(a^0, X^{e_m - 1})}.$$

Mostowski's original definition (?):

Definition. The stratification $\mathcal{X}$ is a Lipschitz stratification if for every $1 < c \in \mathcal{R}$ there exists $C \in \mathcal{R}$ such that $\mathcal{X}$ satisfies the Mostowski condition for $(c, 2c^2, 2c^2, 2c, C)$.

• Playing with the constants

PROPOSITION. *The following conditions on $\mathcal{X}$ are equivalent:*

(1) $\mathcal{X}$ *is a Lipschitz stratification (in the sense of Mostowski).*

(2) For every $c \in \mathcal{R}$, *there exists a* $C \in \mathcal{R}$ *such that* $\mathcal{X}$ *satisfies the Mostowski conditions for* (c, c, C, C, C).

(3) For every $c \in \mathcal{R}$, *there exists a* $C \in \mathcal{R}$ *such that* $\mathcal{X}$ *satisfies the Mostowski conditions for* $(c, c, \frac{1}{c}, \frac{1}{c}, C)$.

Note: (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are easy. But, at first glance, (2) $\Rightarrow$ (3) is hardly plausible, because (3) considers much more chains. To show that, we will (already) need "nonarchimedean extrapolation" of the Mostowski condition.

• Nonarchimedean / nonstandard models

Let $V \subseteq \mathcal{R}$ be a proper convex subring.

FACT. *The subring V is a valuation ring of $\mathcal{R}$.*

DEFINITION. The subring V is called **T-convex** if for all definable (no parameters allowed) continuous function $f : \mathcal{R} \longrightarrow \mathcal{R}$,

$$f(V) \subseteq V.$$

Let T_{convex} be the theory of such pairs $(\mathcal{R}, V)$, where V is an additional symbol in the language.

EXAMPLE. Suppose that T is power bounded. Let $\mathcal{R}$ be the Hahn field $\mathbb{R}((t^{\mathbb{Q}}))$. Let V be the convex hull of $\mathbb{R}$ in $\mathcal{R}$, i.e., $V = \mathbb{R}[[t^{\mathbb{Q}}]]$. Then V is T-convex.

Our proof is actually carried out in a suitable model $(\mathcal{R}, V)$ of T_{convex}, using a mixture of techniques in o-minimality and valuation theories.

• Valuative chains

Let val be the valuation map associated with the valuation ring V.

DEFINITION. A **val-chain** is a sequence of points $a^0, \ldots, a^m$ with $a^\ell \in \mathring{X}^{e_\ell}$ and $e_0 > e_1 > \cdots > e_m$ such that, for all $1 \leq \ell \leq m$,

$$\begin{aligned} \mathrm{val}(a^0 - a^\ell) &= \mathrm{valdist}(a^0, X^{e_{\ell-1}-1}) \\ &= \mathrm{valdist}(a^0, X^{e_\ell}) \\ &> \mathrm{valdist}(a^0, X^{e_\ell - 1}). \end{aligned}$$

An **augmented val-chain** is a val-chain $a^0, \ldots, a^m$ together with one more point $a^{00} \in \mathring{X}^{e_0}$ such that

$$\mathrm{val}(a^0 - a^{00}) > \mathrm{valdist}(a^0, X^{e_0 - 1}).$$

DEFINITION. If we replace $>$ with $\geq$ in the two conditions above then the resulting sequence is called a **weak val-chain**.

Note that a "segment" of a (weak) val-chain is a (weak) val-chain.

- **The valuative Mostowski condition**

DEFINITION. The **valuative Mostowski condition** states: for all val-chain (a^i),

- if (a^i) is not augmented then

$$\mathrm{val}(P_{a^0}^{\perp} P_{a^1} \cdots P_{a^m}) \geq \mathrm{val}(a^0 - a^1) - \mathrm{valdist}(a^0, X^{e_m - 1}),$$

- if (a^i) is augmented then

$$\mathrm{val}((P_{a^0} - P_{a^{00}}) P_{a^1} \cdots P_{a^m}) \geq \mathrm{val}(a^0 - a^{00}) - \mathrm{valdist}(a^0, X^{e_m - 1}).$$

Note: we should use the operator norm above, but $\mathrm{val}(M) = \mathrm{val}(\|M\|)$ for a matrix M.

• Valuative Lipschitz stratification

Definition. The stratification $\mathcal{X}$ is a **valuative Lipschitz stratification** if every val-chain satisfies (the corresponding clause of) the valuative Mostowski condition.

Proposition. *The following are equivalent:*

(1) $\mathcal{X}$ is a Lipschitz stratification in the sense of Mostwoski.

(2) $\mathcal{X}$ is a valuative Lipschitz stratification.

(3) Every weak val-chain satisfies the valuative Mostowski condition.

Note: The valuative "(2) $\Rightarrow$ (3)" here implies the quantitative "(2) $\Rightarrow$ (3)" stated before.

• Strategy / main ingredients of the construction

Let X be a definable closed set in $\mathcal{R}$. We shall construct a stratification $\mathcal{Y}$ of X such that

- $\mathcal{Y}$ is definable in $\mathcal{R}$,
- $\mathcal{Y}$ is a valuative Lipschitz stratification in $(\mathcal{R}, V)$.

We start with any stratification $\mathcal{X} = (X^i)$ of X in $\mathcal{R}$.

The desired stratification is obtained by refining the skeletons $\mathring{X}^s$ one after the other, starting with $\mathring{X}^{\dim X}$. Inductively, suppose that

$$\mathring{X}^{s+1}, \ldots, \mathring{X}^{\dim X}$$

have already been constructed. We refine

$$\mathring{X}^s := X \setminus \bigcup_{i>s} \mathring{X}^i$$

by removing closed subsets of dimension less than s in three steps.

- **The three steps**

Step R1:

We partition $\mathring{X}^s$ into "special cells" and remove all such cells of dimension less than s.

Such a cell is essentially a function $f : A \longrightarrow \mathcal{R}^{n-s}$ of "slow growth", more precisely,

$$\mathrm{val}(f(a) - f(a')) \geq \mathrm{val}(a - a'), \quad \text{for all } a, a' \in A.$$

Actually, we cannot cut $\mathring{X}^s$ into such cells directly; but we can achieve such a decomposition modulo certain "uniform rotation" chosen from a **fixed finite set O of orthogonal matrices**, using a result of Kurdyka / Parusinski / Pawlucki.

Step R2 (the main step):
Consider a sequence $\mathcal{S} = (S^\ell)_{0\le\ell\le m}$, where

$$S^\ell \subseteq \mathring{X}^{e_\ell} \text{ for some } e_0 \ge e_1 > e_2 > \cdots > e_m = s$$

and every S^ℓ is a "special cell" (after a single rotation in O).

There is a subset $Z_{\mathcal{S}} \subseteq S^m$ of dimension less than s such that, once $Z_{\mathcal{S}} \subseteq S^m$ is removed, certain functions associated with $\mathcal{S}$ satisfy certain estimates. There are only finitely many such $Z_{\mathcal{S}}$.

These estimates are all of the form

$$\mathrm{val}(\partial_i f(x)) \ge \mathrm{val}(f(x)) - \mathrm{val}(\zeta_\ell(x)) + \text{ correction terms},$$

where $\zeta_\ell(x)$ is the distance between the tuple $\mathrm{pr}_{\le e_\ell}(x)$ and the subset $\mathcal{R}^{e_\ell} \setminus \mathrm{pr}_{\le e_\ell}(X)$.

Step R3: This step only performs certain cosmetic adjustment. We keep the notation from Step R2 and remove one more set from S^m (again, for each choice of $\mathcal{S}$ and each rotation in O) so that estimates for the functions associated with $\mathcal{S}$ in Step R2 hold on the entire S^m.

This finishes the construction of $\mathring{X}^s$.

Theorem. *The resulting stratification is a valuative Lipschitz stratification of X.*