Families of Isolated Singularities and Three Inspirations from Bernard Teissier

by Terence Gaffney

Cargèse '73

CYCLES EVANESCENTS, SECTIONS PLANES ET CONDITIONS DE WHITNEY

Bernard TEISSIER

Préambule : Le but que l'on se propose ici est la construction d'invariants numériques d'un germe d'hypersurface analytique complexe à singularité isolée, invariants dont la constance dans une petite déformation de l'hypersurface entraîne l'"équisingularité" de cette déformation en un sens très fort (conditions de Whitney). En fait, nous attachons à un germe d'hypersurface $(X_0, x_0) \subset (\mathbb{C}^{n+1}, 0)$ à singularité isolée une suite décroissante d'entiers $\mu^*_{x_0}(X_0) = (\mu^{(n+1)}_{x_0}(X_0), \dots, \mu^{(i)}_{x_0}(X_0), \dots, \mu^{(0)}_{x_0}(X_0))$ où $\mu^{(i)}_{x_0}(X_0)$ est le nombre de cycles évanescents* de l'intersection de (X_0, x_0) avec un i-plan général de $(\mathbb{C}^{n+1}, 0)$. Si $F : X \overset{\sigma}{\rightleftarrows} D = \{t \in \mathbb{C} \,/\, |t| < 1\}$ est une déformation de (X_0, x_0) munie d'une section σ telle que $X - \sigma(\mathbb{D})$ soit lisse sur $\mathbb{D}$, et si "$\mu^{(n+1)}$ constant" (resp. "μ^* constant") signifie que $\mu^{(n+1)}_{\sigma(t)}(X_t) = \mu^{(n+1)}_{x_0}(X_0)$ (resp. avec μ^*), où $X_t = F^{-1}(t)$, pour tout $t \in \mathbb{D}$, notre programme est le suivant :

$$[\mu^{(n+1)} \text{ constant}] \xRightarrow{[C_1]} [\mu^* \text{ constant}] \xRightarrow{[ICI]} [(X - \sigma(\mathbb{D}), \sigma(\mathbb{D})) \text{ satisfait les conditions de Whitney}]$$

[L.R]

[Thom-Mather]

In his Cargèse paper, Teissier introduced

In his Cargèse paper, Teissier introduced

- Integral closure descriptions of equisingularity conditions–Whitney A and B.

In his Cargèse paper, Teissier introduced

- Integral closure descriptions of equisingularity conditions–Whitney A and B.
- Control of integral closure conditions using algebraic invariants–multiplicities of ideals, $\mu^*(X)$.

In his Cargèse paper, Teissier introduced

- Integral closure descriptions of equisingularity conditions–Whitney A and B.
- Control of integral closure conditions using algebraic invariants–multiplicities of ideals, $\mu^*(X)$.
- Geometric descriptions of the algebraic invariants– $\mu^i(X)$ is the Milnor number of a general section of X by an i-plane.

In his Cargèse paper, Teissier introduced

- Integral closure descriptions of equisingularity conditions–Whitney A and B.
- Control of integral closure conditions using algebraic invariants–multiplicities of ideals, $\mu^*(X)$.
- Geometric descriptions of the algebraic invariants– $\mu^i(X)$ is the Milnor number of a general section of X by an i-plane.
- The invariants depend only on the members of the family, not the family.

In his Cargèse paper, Teissier introduced

- Integral closure descriptions of equisingularity conditions–Whitney A and B.
- Control of integral closure conditions using algebraic invariants–multiplicities of ideals, $\mu^*(X)$.
- Geometric descriptions of the algebraic invariants– $\mu^i(X)$ is the Milnor number of a general section of X by an i-plane.
- The invariants depend only on the members of the family, not the family.
- Development: Use integral closure of modules for general analytic spaces, use multiplicity of modules and of pairs of modules for the invariants. Invariants should be independent of family. For ICIS, the invariant controlling the Whitney conditions is $e(mJM(X), \mathcal{O}^p_X)$. ($JM(X)$ denotes the module of partial derivatives of a set of defining equations of X.)(Inv. '92, '96, Sao Carlos, 2002.)

Quick Introduction-Recall : Integral Closure and Equisingularity

Quick Introduction-Recall : Integral Closure and Equisingularity

- $I \subset \mathcal{O}_{X,x}$, an ideal, $\bar{I}$ denotes the integral closure of I, $h \in \mathcal{O}_{X,x}$ is in $\bar{I}$ if as you approach $x \in V(I)$ $h \to 0$ as fast as I does.

Quick Introduction-Recall : Integral Closure and Equisingularity

- $I \subset \mathcal{O}_{X,x}$, an ideal, $\bar{I}$ denotes the integral closure of I, $h \in \mathcal{O}_{X,x}$ is in $\bar{I}$ if as you approach $x \in V(I)$ $h \to 0$ as fast as I does.
- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$.

Quick Introduction-Recall : Integral Closure and Equisingularity

- $I \subset \mathcal{O}_{X,x}$, an ideal, $\bar{I}$ denotes the integral closure of I, $h \in \mathcal{O}_{X,x}$ is in $\bar{I}$ if as you approach $x \in V(I)$ $h \to 0$ as fast as I does.
- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$.
- If $z \in X$ is a smooth point, then $DF(z)$ defines the tangent plane to X at z. ($DF(z)$ is a conormal vector.)

Quick Introduction-Recall : Integral Closure and Equisingularity

- $I \subset \mathcal{O}_{X,x}$, an ideal, $\bar{I}$ denotes the integral closure of I, $h \in \mathcal{O}_{X,x}$ is in $\bar{I}$ if as you approach $x \in V(I)$ $h \to 0$ as fast as I does.
- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$.
- If $z \in X$ is a smooth point, then $DF(z)$ defines the tangent plane to X at z. ($DF(z)$ is a conormal vector.)
- $\partial F/\partial y \in \overline{m_Y J(F)}$ implies that all limiting tangent planes to X at 0 contain Y. ($\partial F/\partial y(z)$ goes to 0 faster than $\partial F/\partial z_i(z)$ go to 0.)

Quick Introduction-Recall : Integral Closure and Equisingularity

- $I \subset \mathcal{O}_{X,x}$, an ideal, $\bar{I}$ denotes the integral closure of I, $h \in \mathcal{O}_{X,x}$ is in $\bar{I}$ if as you approach $x \in V(I)$ $h \to 0$ as fast as I does.
- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$.
- If $z \in X$ is a smooth point, then $DF(z)$ defines the tangent plane to X at z. ($DF(z)$ is a conormal vector.)
- $\partial F/\partial y \in \overline{m_Y J(F)}$ implies that all limiting tangent planes to X at 0 contain Y. ($\partial F/\partial y(z)$ goes to 0 faster than $\partial F/\partial z_i(z)$ go to 0.)
- In fact, the distance between each tangent plane and Y goes to zero as fast as the distance to Y; hence the Whitney conditions hold (Teissier).

Quick Introduction-Recall : Integral Closure and Equisingularity

- $I \subset \mathcal{O}_{X,x}$, an ideal, $\bar{I}$ denotes the integral closure of I, $h \in \mathcal{O}_{X,x}$ is in $\bar{I}$ if as you approach $x \in V(I)$ $h \to 0$ as fast as I does.
- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$.
- If $z \in X$ is a smooth point, then $DF(z)$ defines the tangent plane to X at z. ($DF(z)$ is a conormal vector.)
- $\partial F/\partial y \in \overline{m_Y J(F)}$ implies that all limiting tangent planes to X at 0 contain Y. ($\partial F/\partial y(z)$ goes to 0 faster than $\partial F/\partial z_i(z)$ go to 0.)
- In fact, the distance between each tangent plane and Y goes to zero as fast as the distance to Y; hence the Whitney conditions hold (Teissier).
- If $X = F^{-1}(0)$ is not a hypersurface, and $[M] = [DF(z)]$, then the columns of $[M]$ are generators of $JM(X)$, and the rows of M are a basis of the conormal vectors at a smooth point of X, so $\overline{JM(X)}$ controls limits of tangent hyperplanes at singular points of X.

Passing from generic conditions to the total space: Principle of Specialization of Integral Dependence

Passing from generic conditions to the total space: Principle of Specialization of Integral Dependence

I.3 <u>Théorème</u> (Principe de spécialisation de la dépendance intégrale) : Soit $F : (X,x) \to (S,s)$ un germe de morphisme plat entre espaces de Cohen-Macaulay réduits. Soit I un $\mathcal{O}_X$-Idéal (pour un représentant de F) tel que :

1) Le sous-espace Y de X défini par I soit fini au-dessus de S par F, i.e. $F : Y \to S$ est fini.
2) La multiplicité $e(I.\mathcal{O}_{X_{s'}})$ est indépendante de $s' \in S$ (ici, $e(I.\mathcal{O}_{X_{s'}})$ est la somme des multiplicités des idéaux primaires induits par I dans $\mathcal{O}_{X_{s'}}$. (Chacun des idéaux primaires est primaire dans $\mathcal{O}_{X_{s'}}$ pour un idéal maximal correspondant à un point de $X_{s'} \cap Y$.)

Alors, pour tout représentant assez petit de $F : X \to S$, les conditions suivantes sont équivalentes, pour une fonction $h \in \Gamma(X,\mathcal{O}_X)$:

1) il existe un ouvert analytique dense U de S tel que pour tout $s' \in U$, $h.\mathcal{O}_{X_{s'}} \in \overline{I.\mathcal{O}_{X_{s'}}}$.
2) $h \in \overline{I}$ Idéal cohérent "clôture intégrale de I" de $h \in \Gamma(X,\overline{I})$ [et donc $h.\mathcal{O}_{X_s} \in \overline{I.\mathcal{O}_{X_s}}$ pour tout $s \in S$].

(II Resolution Simultanee et Cycles Evanescents app. 1, 1980)

Principle of Specialization of Integral Dependence

Principle of Specialization of Integral Dependence

- Key points

Principle of Specialization of Integral Dependence

- Key points
- X, X_S, S equidimensional.

Principle of Specialization of Integral Dependence

- Key points
- X, X_S, S equidimensional.
- $e(I \cdot X_s)$ independent of s.

Principle of Specialization of Integral Dependence

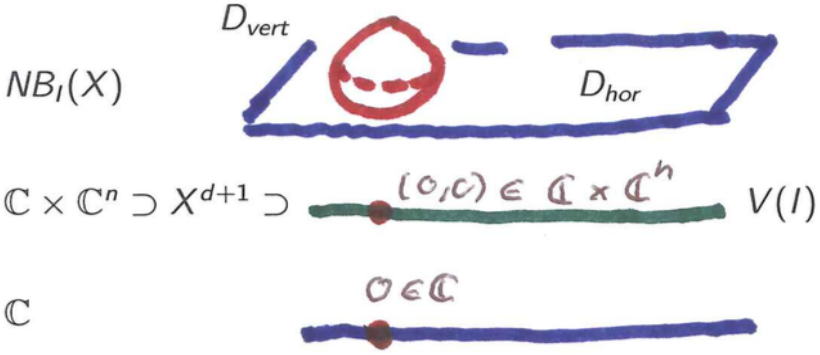
- Key points
- X, X_S, S equidimensional.
- $e(I \cdot X_s)$ independent of s.
- $h \in \bar{I} \cdot X_s$, for s in a Zariski open and every where dense subset of S,

Principle of Specialization of Integral Dependence

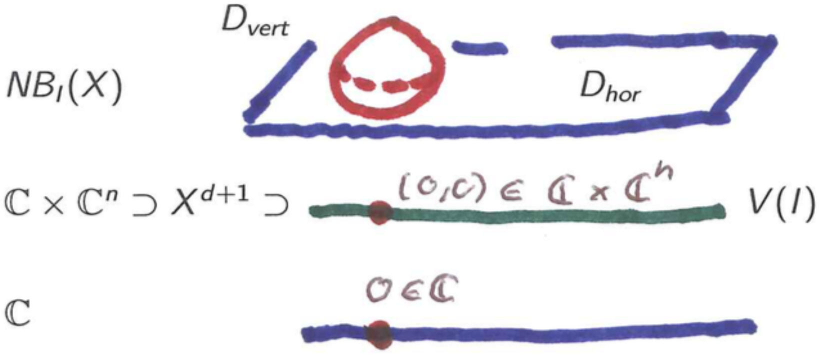
- Key points
- X, X_S, S equidimensional.
- $e(I \cdot X_s)$ independent of s.
- $h \in \bar{I} \cdot X_s$, for s in a Zariski open and every where dense subset of S,
- then $h \in \bar{I} \cdot X$, so fiber wise control gives control on the total space.

Ingredients of the Proof: PSID Schematic

Ingredients of the Proof: PSID Schematic

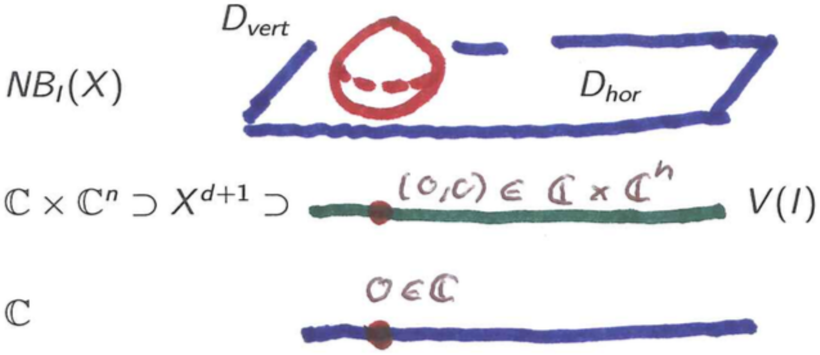


Ingredients of the Proof: PSID Schematic



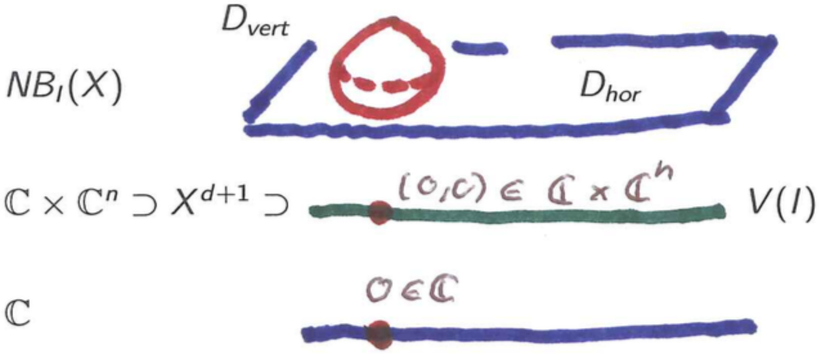
- $NB_I(X), \pi$ is the normalization of the blow-up of X by I. D the exceptional divisor.

Ingredients of the Proof: PSID Schematic



- $NB_I(X), \pi$ is the normalization of the blow-up of X by I. D the exceptional divisor.
- $h \in \overline{I} \cdot X_s$, for s in a Z-open and dense subset of S,implies h vanishes to the desired order on D_{Hor}. ($h \in \overline{I}$ iff $h \circ \pi \in \pi^*(I)$.)

Ingredients of the Proof: PSID Schematic



- $NB_I(X), \pi$ is the normalization of the blow-up of X by I. D the exceptional divisor.
- $h \in \overline{I} \cdot X_s$, for s in a Z-open and dense subset of S,implies h vanishes to the desired order on D_{Hor}. ($h \in \overline{I}$ iff $h \circ \pi \in \pi^*(I)$.)
- The Rees theorem, and upper-semicontinuity of $e(I_s)$ imply I has a reduction with d elements, so D_{vert} is empty.

Development: Kleiman-Thorup Theorem

Development: Kleiman-Thorup Theorem

- X the germ of a reduced analytic space of pure dimension d.

Development: Kleiman-Thorup Theorem

- X the germ of a reduced analytic space of pure dimension d.
- F a free $\mathcal{O}_X$-module,

Development: Kleiman-Thorup Theorem

- X the germ of a reduced analytic space of pure dimension d.
- F a free $\mathcal{O}_X$-module,
- $M \subset N \subset F$ two nested submodules with $M \neq N$, M and N are generically equal and free of rank e. Set $r := d + e - 1$.

Development: Kleiman-Thorup Theorem

- X the germ of a reduced analytic space of pure dimension d.
- F a free $\mathcal{O}_X$-module,
- $M \subset N \subset F$ two nested submodules with $M \neq N$, M and N are generically equal and free of rank e. Set $r := d + e - 1$.
- Set $C := \mathrm{Projan}(\mathcal{R}(M))$ where $\mathcal{R}(M) \subset \mathit{Sym}\,\mathcal{F}$ is the subalgebra induced by M in the symmetric algebra on F. Let $c\colon C \to X$ denote the structure map. Let W be the closed set in X where N is not integral over M, and set $E := c^{-1}W$.

Development: Kleiman-Thorup Theorem

- X the germ of a reduced analytic space of pure dimension d.
- F a free $\mathcal{O}_X$-module,
- $M \subset N \subset F$ two nested submodules with $M \neq N$, M and N are generically equal and free of rank e. Set $r := d + e - 1$.
- Set $C := \mathrm{Projan}(\mathcal{R}(M))$ where $\mathcal{R}(M) \subset Sym\,\mathcal{F}$ is the subalgebra induced by M in the symmetric algebra on F. Let $c\colon C \to X$ denote the structure map. Let W be the closed set in X where N is not integral over M, and set $E := c^{-1}W$.

Development: Kleiman-Thorup Theorem

- X the germ of a reduced analytic space of pure dimension d.
- F a free $\mathcal{O}_X$-module,
- $M \subset N \subset F$ two nested submodules with $M \neq N$, M and N are generically equal and free of rank e. Set $r := d + e - 1$.
- Set $C := \mathrm{Projan}(\mathcal{R}(M))$ where $\mathcal{R}(M) \subset Sym\,\mathcal{F}$ is the subalgebra induced by M in the symmetric algebra on F. Let $c\colon C \to X$ denote the structure map. Let W be the closed set in X where N is not integral over M, and set $E := c^{-1}W$.

Theorem

(JA '94) If N is not integral over M, then E has dimension $r-1$, the maximum possible.

Application

Application

- Suppose $Y^k, 0 \subset X^{d+k}, 0$, Y^k smooth, $\underline{y}$ coordinates on Y, $I(Y) = m_Y$

Application

- Suppose $Y^k, 0 \subset X^{d+k}, 0$, Y^k smooth, $\underline{y}$ coordinates on Y, $I(Y) = m_Y$
- Set $M = m_Y JM(X)$, $N = M + \mathbb{C}\{\partial f/\partial \underline{y}\}$, then $Projan(\mathcal{R}(M)) = B_{m_Y}(C(X))$, $M = N$ off Y.

Application

- Suppose $Y^k, 0 \subset X^{d+k}, 0$, Y^k smooth, $\underline{y}$ coordinates on Y, $I(Y) = m_Y$
- Set $M = m_Y JM(X)$, $N = M + \mathbb{C}\{\partial f/\partial \underline{y}\}$, then $Projan(\mathcal{R}(M)) = B_{m_Y}(C(X))$, $M = N$ off Y.
- Let E denote the exceptional divisor of $B_{m_Y}(C(X))$.

Application

- Suppose $Y^k, 0 \subset X^{d+k}, 0$, Y^k smooth, $\underline{y}$ coordinates on Y, $I(Y) = m_Y$
- Set $M = m_Y JM(X)$, $N = M + \mathbb{C}\{\partial f/\partial \underline{y}\}$, then $Projan(\mathcal{R}(M)) = B_{m_Y}(C(X))$, $M = N$ off Y.
- Let E denote the exceptional divisor of $B_{m_Y}(C(X))$.
- Theorem (Teissier, Larabida) If the fibers of E, the exceptional divisor of $B_{m_Y}(C(X))$ over Y, have the same dimension, then the Whitney conditions hold along Y.

Application

- Suppose $Y^k, 0 \subset X^{d+k}, 0$, Y^k smooth, $\underline{y}$ coordinates on Y, $I(Y) = m_Y$
- Set $M = m_Y JM(X)$, $N = M + \mathbb{C}\{\partial f/\partial \underline{y}\}$, then $Projan(\mathcal{R}(M)) = B_{m_Y}(C(X))$, $M = N$ off Y.
- Let E denote the exceptional divisor of $B_{m_Y}(C(X))$.
- Theorem (Teissier, Larabida) If the fibers of E, the exceptional divisor of $B_{m_Y}(C(X))$ over Y, have the same dimension, then the Whitney conditions hold along Y.
- Proof: If the Whitney conditions fail along Y, they do so on a proper closed subset $S \subset Y$. Then S is the set where $\overline{M} \neq \overline{N}$. (Inv.'92) By the K-T theorem there must be a component of E over S, so the fibers of E have larger dimension over points in S than over the generic point of Y.

Application

- Suppose $Y^k, 0 \subset X^{d+k}, 0$, Y^k smooth, $\underline{y}$ coordinates on Y, $I(Y) = m_Y$
- Set $M = m_Y JM(X)$, $N = M + \mathbb{C}\{\partial f/\partial \underline{y}\}$, then $Projan(\mathcal{R}(M)) = B_{m_Y}(C(X))$, $M = N$ off Y.
- Let E denote the exceptional divisor of $B_{m_Y}(C(X))$.
- Theorem (Teissier, Larabida) If the fibers of E, the exceptional divisor of $B_{m_Y}(C(X))$ over Y, have the same dimension, then the Whitney conditions hold along Y.
- Proof: If the Whitney conditions fail along Y, they do so on a proper closed subset $S \subset Y$. Then S is the set where $\overline{M} \neq \overline{N}$. (Inv.'92) By the K-T theorem there must be a component of E over S, so the fibers of E have larger dimension over points in S than over the generic point of Y.
- For ICIS, use multiplicity of $mJM(X)$ to control the dimension of the fibers of E. What do we do if the multiplicity is not defined? Try $e(JM(X), N)$, for N related to the geometry of X.

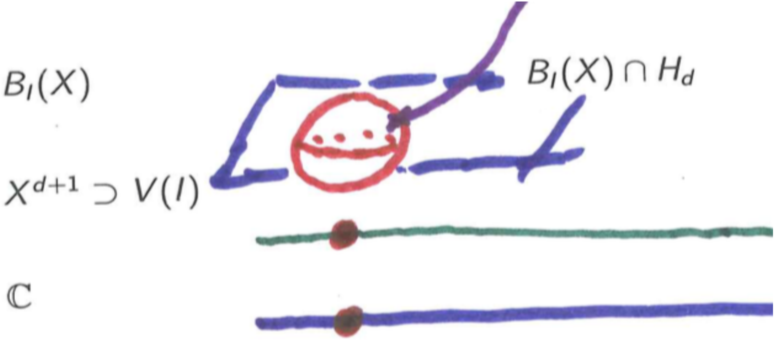
Inspiration 3: LaRabida–1981

Soient $f : (X,0) \to (\mathbb{D},0)$ un morphisme d'espaces réduits, I un idéal de $\mathcal{O}_X$ définissant un sous-espace $Y \subset X$ tel que $f|Y : Y \to \mathbb{D}$ soit fini, $p : X' \to X$ l'éclatement de Y, $D_{vert.}$ la réunion des composantes du diviseur exceptionnel D (non nécessairement réduites) dont l'image ensembliste par p est 0, $\deg D_{vert.} = \deg(\mathcal{O}_{D_{vert.}}(1))$. Pour tout représentant suffisamment petit du germe de f en 0, on a l'égalité

$$\deg D_{vert.} = e(I \cdot \mathcal{O}_{X(0)}) - e(I \cdot \mathcal{O}_{X(s)}) \quad (\text{pour } s \neq 0) \quad .$$

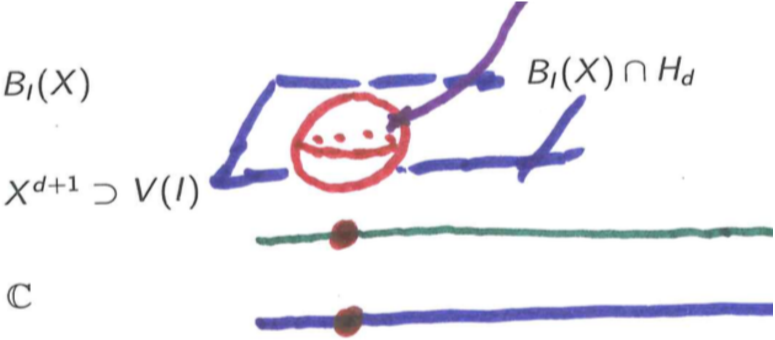
En particulier, on a "$e(I.\mathcal{O}_{X(s)})$ est indépendant de $s \in \mathbb{D}$" si et seulement si $\dim p^{-1}(0) = \dim X - 2$.

Schematic: Degree of D_{vert} over $\mathbb{C}$



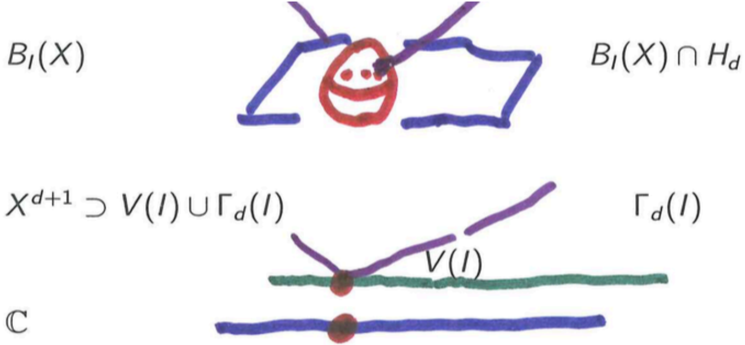
- $H_d = \mathbf{C}^{n+1} \times h_d$, h_d a generic plane of codimension d.

Schematic: Degree of D_{vert} over $\mathbb{C}$



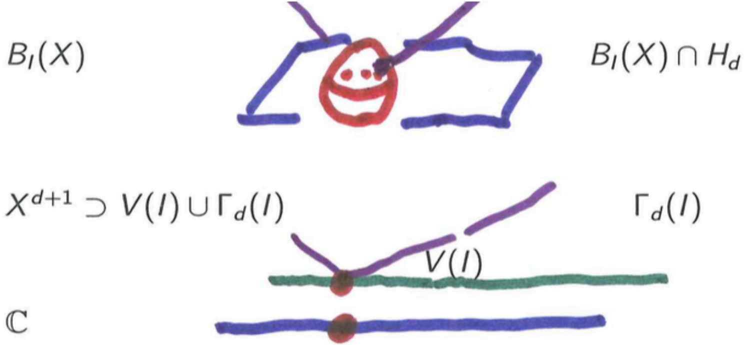
- $H_d = \mathbf{C}^{n+1} \times h_d$, h_d a generic plane of codimension d.
- Degree of D_{vert} is the number of sheets of $B_I(X) \cap H_d$ over $\mathbf{C}$.

$Deg(D_{vert}) = mult_{\mathbb{C}}(\Gamma_d(I)) = \Delta e(I_y, \mathcal{O}_{X,y})$



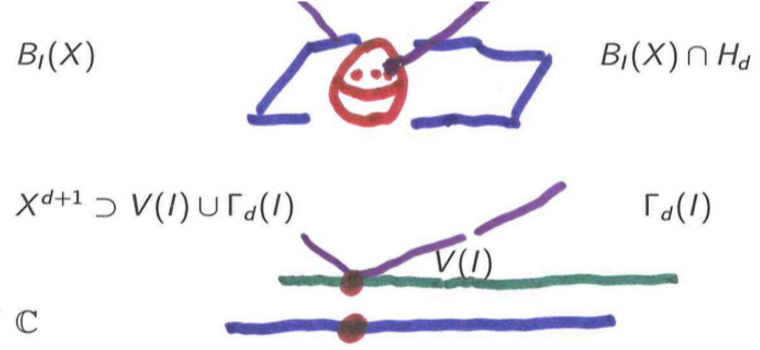
- Polar variety of I of codimension d is the projection of $B_I(X) \cap H_d$ to X.

$Deg(D_{vert}) = mult_{\mathbb{C}}(\Gamma_d(I)) = \Delta e(I_y, \mathcal{O}_{X,y})$



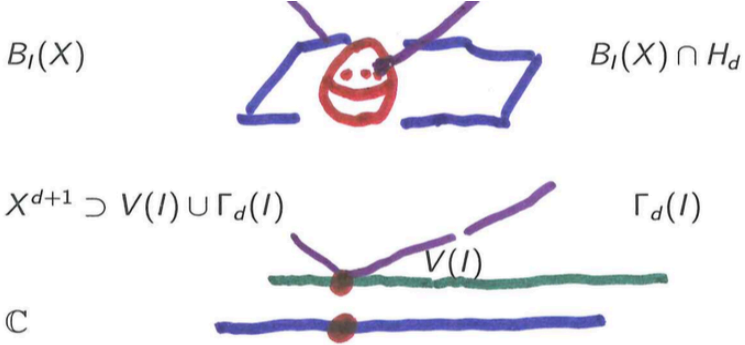
- Polar variety of I of codimension d is the projection of $B_I(X) \cap H_d$ to X.
- (Polar variety of X is the polar variety of $JM(X)$. Polar varieties of X defined in LaRabida.)

$$Deg(D_{vert}) = mult_{\mathbb{C}}(\Gamma_d(I)) = \Delta e(I_y, \mathcal{O}_{X,y})$$



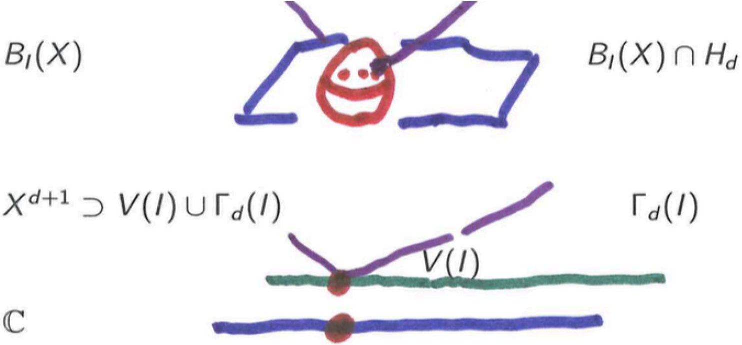
- Polar variety of I of codimension d is the projection of $B_I(X) \cap H_d$ to X.
- (Polar variety of X is the polar variety of $JM(X)$. Polar varieties of X defined in LaRabida.)
- $\Gamma_d(I)$ defined using d generic generators $\underline{h}$.

$Deg(D_{vert}) = mult_{\mathbb{C}}(\Gamma_d(I)) = \Delta e(I_y, \mathcal{O}_{X,y})$



- Polar variety of I of codimension d is the projection of $B_I(X) \cap H_d$ to X.
- (Polar variety of X is the polar variety of $JM(X)$. Polar varieties of X defined in LaRabida.)
- $\Gamma_d(I)$ defined using d generic generators $\underline{h}$.
- $V(\underline{h}_s) = V(I_s) \cup \Gamma_d(I)(s)$. At points x of $V(I_s)$, $e(I_s, x) = e(\underline{h}_s, x)$.

$$Deg(D_{vert}) = mult_{\mathbb{C}}(\Gamma_d(I)) = \Delta e(I_y, \mathcal{O}_{X,y})$$



- Polar variety of I of codimension d is the projection of $B_I(X) \cap H_d$ to X.
- (Polar variety of X is the polar variety of $JM(X)$. Polar varieties of X defined in LaRabida.)
- $\Gamma_d(I)$ defined using d generic generators $\underline{h}$.
- $V(\underline{h}_s) = V(I_s) \cup \Gamma_d(I)(s)$. At points x of $V(I_s)$, $e(I_s, x) = e(\underline{h}_s, x)$.
- $e(\underline{h}_s, s)$ independent of s, so $\Delta(e(I)) = mult_{\mathbf{C}}(\Gamma_d(I))$.

Implications for equisingularity

- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$, $J_z(F) = (\partial F/\partial z_1, \ldots \partial F/\partial z_{n+1})$, $f_y = F(y, \_)$.

Implications for equisingularity

- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$, $J_z(F) = (\partial F/\partial z_1, \ldots \partial F/\partial z_{n+1})$, $f_y = F(y, \_)$.
- Consider $B_{J_z(F)}(X)$; its fiber over $(y, z) \in X$ consists of tangent planes to X_y at z.

Implications for equisingularity

- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$, $J_z(F) = (\partial F/\partial z_1, \ldots \partial F/\partial z_{n+1})$, $f_y = F(y, \_)$.
- Consider $B_{J_z(F)}(X)$; its fiber over $(y, z) \in X$ consists of tangent planes to X_y at z.
- Suppose $\mu(f_y)$ changes at $y = 0$.

Implications for equisingularity

- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$, $J_z(F) = (\partial F/\partial z_1, \ldots \partial F/\partial z_{n+1})$, $f_y = F(y, \_)$.
- Consider $B_{J_z(F)}(X)$; its fiber over $(y, z) \in X$ consists of tangent planes to X_y at z.
- Suppose $\mu(f_y)$ changes at $y = 0$.
- Then $e(J(f_y))$ changes, so there must be a vertical component of $D \subset B_{J_z(F)}(X)$ over zero.

Implications for equisingularity

- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$, $J_z(F) = (\partial F/\partial z_1, \ldots \partial F/\partial z_{n+1})$, $f_y = F(y, \_)$.
- Consider $B_{J_z(F)}(X)$; its fiber over $(y, z) \in X$ consists of tangent planes to X_y at z.
- Suppose $\mu(f_y)$ changes at $y = 0$.
- Then $e(J(f_y))$ changes, so there must be a vertical component of $D \subset B_{J_z(F)}(X)$ over zero.
- In other words, a change in the topology of the smoothing of X_y (the Milnor fiber) causes a change in the infinitesimal geometry of X.

Implications for equisingularity

- Set-up: $Y = \mathbb{C} \times \{0\} \subset X^{n+1}, 0 \subset \mathbb{C} \times \mathbb{C}^{n+1}$, $X = F^{-1}(0)$, $m_Y = I(Y)$, $Y = S(X)$, $J_z(F) = (\partial F/\partial z_1, \ldots \partial F/\partial z_{n+1})$, $f_y = F(y, \_)$.
- Consider $B_{J_z(F)}(X)$; its fiber over $(y, z) \in X$ consists of tangent planes to X_y at z.
- Suppose $\mu(f_y)$ changes at $y = 0$.
- Then $e(J(f_y))$ changes, so there must be a vertical component of $D \subset B_{J_z(F)}(X)$ over zero.
- In other words, a change in the topology of the smoothing of X_y (the Milnor fiber) causes a change in the infinitesimal geometry of X.
- Moral: Families of sets are part of larger landscapes, and changes in the topology of the landscape affects the infinitesimal geometry of the family.

Development: The Multiplicity Polar Theorem (MPT)

Development: The Multiplicity Polar Theorem (MPT)

- $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y,with equidimensional fibers, Y smooth.

Development: The Multiplicity Polar Theorem (MPT)

- $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y,with equidimensional fibers, Y smooth.
- $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.

Development: The Multiplicity Polar Theorem (MPT)

- $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y,with equidimensional fibers, Y smooth.
- $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.
- $\Delta(e(M,N)) = e(M(0),N(0),\mathcal{O}_{X(0)},0) - e(M(y),N(y),\mathcal{O}_{X(y)},(y,x))$ is the change in the multiplicity of the pair (M,N) as the parameter changes from y to 0.

Development: The Multiplicity Polar Theorem (MPT)

- $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y,with equidimensional fibers, Y smooth.
- $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.
- $\Delta(e(M,N)) = e(M(0), N(0), \mathcal{O}_{X(0)}, 0) - e(M(y), N(y), \mathcal{O}_{X(y)}, (y,x))$ is the change in the multiplicity of the pair (M, N) as the parameter changes from y to 0.
- Formula (MPT, '02):

$$\Delta(e(M,N)) = mult_y \Gamma_d(M) - mult_y \Gamma_d(N)$$

Development: The Multiplicity Polar Theorem (MPT)

- $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y,with equidimensional fibers, Y smooth.
- $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.
- $\Delta(e(M,N)) = e(M(0), N(0), \mathcal{O}_{X(0)}, 0) - e(M(y), N(y), \mathcal{O}_{X(y)}, (y,x))$ is the change in the multiplicity of the pair (M,N) as the parameter changes from y to 0.
- Formula (MPT, '02):

$$\Delta(e(M,N)) = mult_y \Gamma_d(M) - mult_y \Gamma_d(N)$$

- With the appropriate N we can use this for a numerical criterion for the Whitney conditions to hold for the open stratum at $X, 0$.

Development: The Multiplicity Polar Theorem (MPT)

- $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y,with equidimensional fibers, Y smooth.
- $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.
- $\Delta(e(M, N)) = e(M(0), N(0), \mathcal{O}_{X(0)}, 0) - e(M(y), N(y), \mathcal{O}_{X(y)}, (y, x))$ is the change in the multiplicity of the pair (M, N) as the parameter changes from y to 0.
- Formula (MPT, '02):

$$\Delta(e(M, N)) = mult_y \Gamma_d(M) - mult_y \Gamma_d(N)$$

- With the appropriate N we can use this for a numerical criterion for the Whitney conditions to hold for the open stratum at $X, 0$.
- Choosing N should reflect a choice of "landscape".

Determinantal singularities

- Given M, a $(n+k,n)$ matrix, with entries in $\mathcal{O}_q$; view M as a map from $\mathbf{C}^q \to Hom(\mathbf{C}^n,\mathbf{C}^{n+k})$. Assume M is transverse to the rank stratification of $Hom(\mathbf{C}^n,\mathbf{C}^{n+k})$ on $\mathbf{C}^q - 0$.

Determinantal singularities

- Given M, a $(n+k,n)$ matrix, with entries in $\mathcal{O}_q$; view M as a map from $\mathbf{C}^q \to Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$. Assume M is transverse to the rank stratification of $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ on $\mathbf{C}^q - 0$.
- $X_M := V(I_M)$, I_M generated by the maximal minors of M. X_M is determinantal i.e. $\text{codim}(X_M)$ is as small as possible.

Determinantal singularities

- Given M, a $(n+k,n)$ matrix, with entries in $\mathcal{O}_q$; view M as a map from $\mathbf{C}^q \to Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$. Assume M is transverse to the rank stratification of $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ on $\mathbf{C}^q - 0$.
- $X_M := V(I_M)$, I_M generated by the maximal minors of M. X_M is determinantal i.e. $\operatorname{codim}(X_M)$ is as small as possible.
- If $q < 2(2+k)$ then X_M has a smoothing.

Determinantal singularities

- Given M, a $(n+k,n)$ matrix, with entries in $\mathcal{O}_q$; view M as a map from $\mathbf{C}^q \to Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$. Assume M is transverse to the rank stratification of $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ on $\mathbf{C}^q - 0$.
- $X_M := V(I_M)$, I_M generated by the maximal minors of M. X_M is determinantal i.e. $\mathrm{codim}(X_M)$ is as small as possible.
- If $q < 2(2+k)$ then X_M has a smoothing.
- We fix the class of deformations and fix a unique smoothing by only considering deformations of X_M which come from deformations of the entries of M. Geometric meaning of invariants depends on the smoothing.

Determinantal singularities

- Given M, a $(n+k, n)$ matrix, with entries in $\mathcal{O}_q$; view M as a map from $\mathbf{C}^q \to Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$. Assume M is transverse to the rank stratification of $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ on $\mathbf{C}^q - 0$.
- $X_M := V(I_M)$, I_M generated by the maximal minors of M. X_M is determinantal i.e. $\text{codim}(X_M)$ is as small as possible.
- If $q < 2(2+k)$ then X_M has a smoothing.
- We fix the class of deformations and fix a unique smoothing by only considering deformations of X_M which come from deformations of the entries of M. Geometric meaning of invariants depends on the smoothing.
- Deformations of the entries of M induce deformations of the generators of I; first order deformations define the module $N(X_M)$. Generators of $N(X_M)$ are tuples of minors of M of size $n-1$.

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.
- Stability implies the polar varieties of Σ are the polar varieties of N_U.

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.
- Stability implies the polar varieties of Σ are the polar varieties of N_U.
- Universality implies $\Gamma_i(N(X_M)) = M^*\Gamma_i(N_U)$.

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.
- Stability implies the polar varieties of Σ are the polar varieties of N_U.
- Universality implies $\Gamma_i(N(X_M)) = M^*\Gamma_i(N_U)$.
- Together they imply if $\tilde{M}$ defines a smoothing $\tilde{X}$ of X^d_M, then

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.
- Stability implies the polar varieties of Σ are the polar varieties of N_U.
- Universality implies $\Gamma_i(N(X_M)) = M^*\Gamma_i(N_U)$.
- Together they imply if $\tilde{M}$ defines a smoothing $\tilde{X}$ of X^d_M, then
- $$mult_{\mathbf{C}}\Gamma_d(N(\tilde{X}_{\tilde{M}})) = M(\mathbf{C}^q) \cdot \Gamma_d(\Sigma)$$

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.
- Stability implies the polar varieties of Σ are the polar varieties of N_U.
- Universality implies $\Gamma_i(N(X_M)) = M^*\Gamma_i(N_U)$.
- Together they imply if $\tilde{M}$ defines a smoothing $\tilde{X}$ of X^d_M, then
- $$mult_{\mathbf{C}}\Gamma_d(N(\tilde{X}_{\tilde{M}})) = M(\mathbf{C}^q) \cdot \Gamma_d(\Sigma)$$
- For maximal minors, $M(\mathbf{C}^q) \cdot \Gamma_d(\Sigma)$ computed in terms of the entries of M in [G-Rangachev], MathArxiv 1501.00201.

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.
- Stability implies the polar varieties of Σ are the polar varieties of N_U.
- Universality implies $\Gamma_i(N(X_M)) = M^*\Gamma_i(N_U)$.
- Together they imply if $\tilde{M}$ defines a smoothing $\tilde{X}$ of X^d_M, then
- $$mult_{\mathbf{C}}\Gamma_d(N(\tilde{X}_{\tilde{M}})) = M(\mathbf{C}^q)\cdot\Gamma_d(\Sigma)$$
- For maximal minors, $M(\mathbf{C}^q)\cdot\Gamma_d(\Sigma)$ computed in terms of the entries of M in [G-Rangachev], MathArxiv 1501.00201.
- As a corollary of the MPT, $e(mJM(X_{M(y)}), N(X_{M(y)})) + M(y)(\mathbf{C}^q)\cdot\Gamma_d(\Sigma)$ controls the Whitney conditions for the open stratum of X_M along Y.

Properties of $N(X)$

- N is universal. If the entries of M are coordinates on $Hom(\mathbf{C}^n, \mathbf{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.
- Stability implies the polar varieties of Σ are the polar varieties of N_U.
- Universality implies $\Gamma_i(N(X_M)) = M^*\Gamma_i(N_U)$.
- Together they imply if $\tilde{M}$ defines a smoothing $\tilde{X}$ of X^d_M, then
- $$mult_{\mathbf{C}}\Gamma_d(N(\tilde{X}_{\tilde{M}})) = M(\mathbf{C}^q) \cdot \Gamma_d(\Sigma)$$
- For maximal minors, $M(\mathbf{C}^q) \cdot \Gamma_d(\Sigma)$ computed in terms of the entries of M in [G-Rangachev], MathArxiv 1501.00201.
- As a corollary of the MPT, $e(mJM(X_{M(y)}), N(X_{M(y)})) + M(y)(\mathbf{C}^q) \cdot \Gamma_d(\Sigma)$ controls the Whitney conditions for the open stratum of X_M along Y.
- $e(JM(X_{M(y)}), N(X_{M(y)})) + M(y)(\mathbf{C}^q) \cdot \Gamma_d(\Sigma) = (-1)^d\chi(X_{s,y}) + (-1)^{d-1}\chi((X \cap H)_{s,y}), X_{s,y}$ a smoothing of $X(y)$.

- Joyeux Anniversaire,

  Bernard !

- Joyeux Anniversaire,

  Bernard !

- Merci beaucoup pour tous les dons !

- # Joyeux Anniversaire,

  # Bernard !

- # Merci beaucoup pour tous les dons !

- *Cycles évanescents, sections planes et conditions de Whitney,* in "Singularités à Cargèse," Astérisque **7–8** (1973), 285–362

- # Joyeux Anniversaire,

  # Bernard !

- # Merci beaucoup pour tous les dons !

- *Cycles évanescents, sections planes et conditions de Whitney,* in "Singularités à Cargèse," Astérisque **7–8** (1973), 285–362
- *Résolution simultanée et cycles évanescents,* in "Sém. sur les singularités des surfaces." Springer Lecture Notes 777 1980 82–146

- # Joyeux Anniversaire,

  # Bernard !

- # Merci beaucoup pour tous les dons !

- *Cycles évanescents, sections planes et conditions de Whitney,* in "Singularités à Cargèse," Astérisque **7–8** (1973), 285–362
- *Résolution simultanée et cycles évanescents,* in "Sém. sur les singularités des surfaces." Springer Lecture Notes 777 1980 82–146
- *Multiplicités polaires, sections planes, et conditions de Whitney,* in "Proc. La Rábida, 1981." J. M. Aroca, R. Buchweitz, M. Giusti and M. Merle (eds.) Springer Lecture Notes 961 1982 314–491

- # Joyeux Anniversaire,

  # Bernard !

- # Merci beaucoup pour tous les dons !

- *Cycles évanescents, sections planes et conditions de Whitney,* in "Singularités à Cargèse," Astérisque **7–8** (1973), 285–362
- *Résolution simultanée et cycles évanescents,* in "Sém. sur les singularités des surfaces." Springer Lecture Notes 777 1980 82–146
- *Multiplicités polaires, sections planes, et conditions de Whitney,* in "Proc. La Rábida, 1981." J. M. Aroca, R. Buchweitz, M. Giusti and M. Merle (eds.) Springer Lecture Notes 961 1982 314–491
- And all the others!