ALGEBRAIC GEOMETRY, THEORY OF SINGULARITIES, AND CONVEX GEOMETRY

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There is also an intermediate version of these theories. It provides such a relation in the framework of **spherical varieties**.

Newton polyhedra

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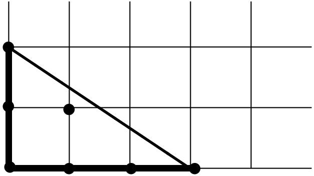
Example. Let P be $y^2 + a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_0 \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$. Then $\Delta(P)$ is



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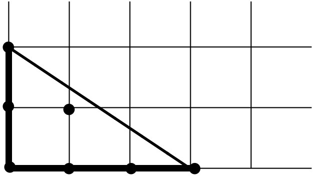

and $s(P) = \{(0,0), (0,1), (0,2), (0,3), (2,0)\}$.

Discrete invariants of $X \subset (\mathbb{C}^*)^n$ defined by a generic system of equations $P_1(x) = \cdots = P_k = 0$ with fixed support $s(P_i)$ depend only on Newton polyhedra $\Delta(P_1), \ldots, \Delta(P_k)$ of $P_1, \ldots, P_k$.

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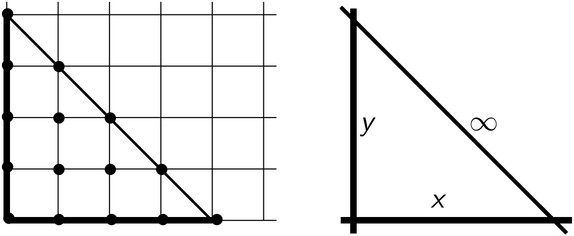
Toy geometric application. The invariants 1)-3) are related:
$\chi(\bar{X}) = \chi(X) + \#A(X) = 2 - 2g(X)$.
It implies the **Pick formula** for an integral polygon Δ:
$V(\Delta) = \#((\Delta \setminus \partial\Delta) \bigcap \mathbb{Z}^2) + 1/2\#\partial(\Delta \bigcap \mathbb{Z}^2) - 1$.

Toric varieties and the combinatorics of polyhedra



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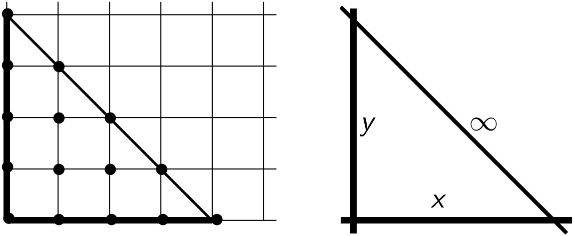

Toric variety is a normal connected n-dimensional algebraic variety M on which an $(\mathbb{C}^*)^n$ acts algebraically and has one orbit isomorphic to $(\mathbb{C}^*)^n$. Under the action of $(\mathbb{C}^*)^n$, M is broken up into a finite number of orbits isomorphic to tori of different dimensions. To every Newton polyhedron Δ we can associate a compact projective toric variety M_Δ in such a way that every k-dimensional face $\Gamma \subset \Delta$ corresponds to a complex k-dimensional orbit $O_\Gamma \subset M_\Delta$. If $\Gamma_1 \subset \Gamma_2$ then $O_{\Gamma_1} \subset \bar{O}_{\Gamma_2}$.

Simple polyhedra and quasismooth toric varieties

A polyhedron is **simple** if it is an intersection of half-spaces in general position. An n-dimensional simple polyhedron about each vertex has the same structure as the positive orthant in $\mathbb{R}^n$ near the origin. In particular, each vertex of a simple n-dimensional polyhedron is incident with n edges, and any k of these edges belong to one k-dimensional face containing the vertex.

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The F-**vector** of a simple n-dimensional polyhedron is the vector $(F_0, \ldots, F_n)$ where F_k is the number of k-dimensional faces of the polyhedron. The necessary and sufficient conditions for a vector to be F-vector of a simple n-dimensional polyhedron were conjectured by *McMullen*.

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Simple polyhedra correspond to **guasismooth toric varieties**. Using topology and algebraic geometry of such varieties *Stanley, Billera* and *Lee* proved McMullen's conjecture.

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Theorem (*Nikulin*). *The average number of l-dimensional faces of a k-dimensional face of a simple n-dimensional polyhedron for $0 \leq l < k \leq (n+1)/2$ is $\leq f(l,k,n)$ where f is an explicit function. If $n \to \infty$, f tends to the number of l-dimensional faces of a k-dimensional cube.*

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Theorem (*Prokhorov, Kh*). *In a Lobachevsky space of dimension* > 995 *there are no discrete groups generated by reflections with a fundamental polyhedron of finite volume.*

How many solutions in $(\mathbb{C}^*)^n$ has a system of equations $P_1 = \cdots = P_n = 0$ where $P_1, \ldots, P_n$ are generic Laurent polynomials with the fixed supports $A_1, \ldots, A_n \subset \mathbb{Z}^n$?

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Denote by L_{A_i} be the space generated x^m, where $m \in A_i$.
Denote by Δ_i the convex hull of A_i.
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Theorem (*Kouchnirenko*). *If $A_1 = \cdots = A_n = A$ then the number of solutions of the system is equal to the volume $V(\Delta)$ of $\Delta = \Delta_1 = \cdots = \Delta_n$ multiplied by $n!$*

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Theorem (*Bernstein*) (also known as BKK theorem). *The number of solutions of the system is equal to the mixed volume $V(\Delta_1, \ldots, \Delta_n)$ of $\Delta_1, \ldots, \Delta_n$ multiplied ny $n!$*

Mixed volume is a unique function $V(\Delta_1, \dots, \Delta_n)$ on n-tuples of convex bodies in $\Delta_i \subset \mathbb{R}^n$, such that:

1. $V(\Delta, \dots, \Delta)$ is the volume of Δ;
2. V is symmetric;
3. V is multi-linear; for example, $V(\Delta_1' + \Delta_1'', \Delta_2, \dots) = V(\Delta_1', \Delta_2, \dots) + V(\Delta_1'', \Delta_2, \dots)$;

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4. V is nonnegative, i.e. $0 \leq V(\Delta_1, \ldots, \Delta_n)$;
5. $\Delta_1' \subseteq \Delta_1, \ldots, \Delta_n' \subseteq \Delta_n \Rightarrow V(\Delta_1', \ldots, \Delta_n') \leq V(\Delta_1, \ldots, \Delta_n)$;
6. The following **Alexandrov–Fenchel inequality** holds: $V^2(\Delta_1, \Delta_2, \ldots, \Delta_n) \geq$ $V(\Delta_1, \Delta_1, \ldots, \Delta_n)V(\Delta_2, \Delta_2, \ldots, \Delta_n)$;
7. in particular (for $n = 2$, the unite ball Δ_1 and for $\Delta = \Delta_2$) the **isoperimetric inequality** ($\frac{1}{2}$ length of $\partial\Delta)^2 \geq \pi V(\Delta)$ holds.

Intersection index on an irreducible variety X

Let $K(X)$ be the **semigroup of spaces** L of rational functions on X such that: a) $\dim L < \infty$, and b) $L \neq 0$.
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BKK theorem computes the intersection index **for** $X = (\mathbb{C}^*)^n$ and for an n-tuple of **spaces generated by monomials**.

Grothendieck semigroup and group

For **a commutative semigroup** S consider the following equivalence relation:
$a \sim b \Leftrightarrow (\exists c \in S)|(a + c = b + c)$.
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Theorem (*Kh*). *Let $\mathcal{K}$ be the semigroup of finite subsets $\mathbb{Z}^n \subset \mathbb{R}^n$ with respect to addition. Then $\mathrm{Gr}_s(\mathcal{K})$ is the semigroup of convex integral polyhedra in $\mathbb{R}^n$ and $\rho(A)$ is the convex hull $\Delta(A)$ of A.*

We need the following algebraic analog of this theorem.

The Grothendieck semigroup $Gr_s(K(X))$ of $K(X)$

A function $f \in \mathbb{C}(X)$ is **integral over** $L \in K(X)$ if it satisfies an equation $f^m + a_1 f^{m-1} + \cdots + a_m = 0$ with $m > 0$ and $a_i \in L^i$.

The **completion** $\overline{L}$ **of** $L \in K(X)$ is the set of all functions integral over L. The **set** $\overline{L}$ **is a finite-dimensional space**, so $\overline{L} \in K(X)$.

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The index $[L_1, \ldots, L_n]$ can be extended to the Grothendieck group $\mathrm{Gr}(K(X))$ of $K(X)$ and can be considered as **a birationally invariant version of the intersection index of divisors**, which is applicable to non-complete varieties.

Regularization of a semigroup of integral points

For a **semigroup** $S \subset \mathbb{Z}^n$ **of integral points** let:
1) $G(S) \subset \mathbb{Z}^n$ be the group generated by S;
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Theorem (*Kaveh, Kh*) *Let $C' \subset C(S)$ be a strongly convex cone which intersects the boundary (in the topology of the linear space $L(S)$) of the cone $C(S)$ only at the origin. Then there exists a constant $N > 0$ (depending on C') such that any point in the group $G(S)$ which lies in C' and whose distance from the origin is bigger than N belongs to S.*

Semigroup of integral points and its NO body

Let M by a hyperplane in $L(S)$. Let M_k be the affine space parallel to M and intersecting $G(S)$ and $C(S)$ which has distance k from the origin (the distance is normalized in such a way that as values it takes all the non-negative integers k).

The **Hilbert function** H_S **of the semigroup** S **in the codirection** M is defined by $H_S(k) = \#M_k \cap S$.

The Newton–Okounkov body (NO body) of the semigroup S **in the codirection** M is defined by $\Delta(S, M) = C(S) \cap M_1$.

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Theorem (*Kaveh, Kh*)*The function* $H_S(k)$ *grows like* $a_q k^q$ *where* q *is the dimension of the convex body* $\Delta(S)$*, and the q-th growth coefficient* a_q *is equal to the (normalized in the appropriate way) q-dimensional volume of* $\Delta(S)$.

Algebra of almost finite type, its NO body (begging)

Let F be a field of transcendence degree n over $\mathbf{k}$. We deal with the following kinds of graded subalgebras in the algebra $F[t]$:

1. With any subspace $L \subset F$ over $\mathbf{k}$ of finite dimension one associates the **algebra** $A_L = \bigoplus_{k \geq 0} L^k t^k$, where $L^0 = \mathbf{k}$ and L^k is the span of all the products $f_1 \cdots f_k$ with $f_1, \dots, f_k \in L$.
2. An **algebra of almost finite type** is a graded subalgebra in some algebra A_L.

Algebra of almost finite type, its NO body (begging)

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One can construct a $\mathbb{Z}^{n+1}$-valued valuation v_t on $F[t]$ by extending a $\mathbb{Z}^n$-valuation v on F which takes all the values in $\mathbb{Z}^n$.

The **NO body of algebra** A **of almost finite type** is the NO body of the semigroup $S(A) = v_t(A \setminus \{0\})$ projected to the first factor $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.

Algebra and its NO body (continuation)

Theorem (*Kaveh, Kh*). *The Hilbert function $H_A(k)$ of an algebra A of almost finite type grows like $a_q k^q$, where $q = \dim_{\mathbb{R}} \Delta(A)$ and a_q is the (normalized) q-dimensional volume of $\Delta(A)$.*

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One defines a componentwise product of graded subalgebras. Consider the class of graded algebras of almost finite type such that, for $k \gg 0$, all their k-th homogeneous components are non-zero. Let A_1, A_2 be algebras of such kind and put $A_3 = A_1 A_2$. **It is easy to verify the inclusion** $\Delta(A_1) + \Delta(A_2) \subset \Delta(A_3)$.

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Brunn–Minkowsky inequality in convex geometry
$V^{1/n}(\Delta_1) + V^{1/n}(\Delta_2) \leq V^{1/n}(\Delta_1 + \Delta_2)$.

Theorem (*Kaveh, Kh*). $a_n^{1/n}(A_1) + a_n^{1/n}(A_2) \leq a_n^{1/n}(A_3)$.

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1. $[L_1, \dots, L_1]^{1/n} + [L_2, \dots, L_2]^{1/n} \leq [L_3, \dots, L_3]^{1/n}$.
2. Hodge type inequality. For $n = 2$ *we have*
$[L_1, L_1][L_2, L_2] \leq [L_1, L_2]^2$.

An inequality related to Teissier's works

Alexandrov–Fenchel inequality in convex geometry claims that $V^2(\Delta_1, \Delta_2, \dots, \Delta_n) \geq V(\Delta_1, \Delta_1, \dots, \Delta_n)V(\Delta_2, \Delta_2, \dots, \Delta_n)$.

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Let X, $\dim X = n$, be an irreducible variety, let $L_1, \dots, L_n \in K(X)$.
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The Alexandrov–Fenchel inequality in convex geometry follows easily from this theorem via the BKK theorem. This trick has been known. Our elementary proof of the key analogue of the Hodge index inequality which makes all the chain of arguments involved elementary and more natural.

Other results (begginng)

1. Let $\mathbf{K}_a$ be the set of primary ideals of the ring of regular functions at $\mathbf{a} \in X, \dim X = n$. The local intersection index $[L_1, \dots, L_n]_a$ for $L_i \in \mathbf{K}_a$ is equal to the multiplicity at $\mathbf{a}$ of a system $f_1 = \dots = f_n = 0$, where f_i is a generic function from L_i.

A version of Teissier inequality for mixed multiplicities

$[L_1, L_2, \dots, L_n]_a^2 \leq [L_1, L_1, \dots, L_n]_a [L_2, L_2, \dots, L_n]_a$.

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2. Let $C \subset \mathbb{R}^n$ be a strongly convex cone. A compact set $A \subset C$ is called C-**co-convex body** if $C \setminus A$ is convex. One can construct a **theory of** C**- co-convex bodies** analogous to the theory of convex bodies and define the mixed volume $V_C(A_{i_1} \dots, A_{i_n})$ of an n-tuple of C-co-convex bodies $A_{i_1} \dots, A_{i_n})$.

Local geometric Alexandrov–Fenchel inequality

$V_C(A_1, A_2, \dots, A_n)^2 \leq V_C(A_1, A_1, \dots, A_n) V_C(A_2, A_2, \dots, A_n)$.

Other results (continuation)

3. **NO body and Fujita approximation theorem**. One can prove an analogues of Fujita approximation theorem for semigroups of integral points and for graded algebras of almost finite type.
As a corollary one obtains a generalization of the classical Fujita theorem for arbitrary linear series.

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3. **NO body and Fujita approximation theorem**. One can prove an analogues of Fujita approximation theorem for semigroups of integral points and for graded algebras of almost finite type. As a corollary one obtains a generalization of the classical Fujita theorem for arbitrary linear series.

4. **NO body and reductive group action**. Assume that X is equipped with an action of a reductive group G and that one is interested only in G-invariant subspaces $L \in \mathcal{K}(X)$. Then it is possible by choosing an appropriate $\mathbb{Z}^n$-valued valuation v on $\mathbb{C}(X)$ to make all results more precise and explicit.

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If an action is **spherical** one can compute the **arithmetic genus** an some other invariants of a generic complete intersection of G-invariant linear systems.

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CONGRATULATIONS !

BEST WISHES TO BERNARD TEISSIER