Free divisors and rational cuspidal curves

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Singular Landscapes Aussois
25th June 2015
In honor of Bernard Teissier

Reference

Joint work with Gabriel Sticlaru (Bucharest)

Free divisors and rational cuspidal plane curves, arxiv:1504.01242

Nearly free divisors and rational cuspidal curves, arxiv:1505.00666

1. The main characters of our story
   - Rational cuspidal curves
   - Free divisors in $\mathbb{P}^2$
   - Nearly free divisors in $\mathbb{P}^2$
2. Main conjectures and main results
3. Ingredients in the proofs
   - Uli Walther's result
   - Proof in the even degree case
   - A Lefchetz type property for the graded module $N(f)$

Rational cuspidal curves: definition and first examples

Let $C : f = 0$ be a reduced plane curve in the complex projective plane $\mathbb{P}^2$, defined by a degree d homogeneous polynomial f in the graded polynomial ring $S = \mathbb{C}[x, y, z]$.
We say that C is a rational cuspidal curve (r.c.c.) if C is homeomorphic to $\mathbb{P}^1$, i.e. C is irreducible, any singularity of C is unibranch and

$$\mu(C) = (d-1)(d-2),$$

the maximal possible value. Indeed

$$g = \frac{(d-1)(d-2)}{2} - \sum \delta_x$$

with

$$\delta_x = \frac{\mu_x + r_x - 1}{2} = \frac{\mu_x}{2}$$

when $r_x = 1$.

Example (E. Artal Bartolo, T. K. Moe, M. Namba)

1. If $d=1$, then C is a line, e.g. $x=0$;
2. If $d=2$, then C is a smooth conic, e.g. $x^2+y^2+z^2=0$;
3. If $d=3$, then C is a cuspidal cubic, e.g. $x^2y+z^3=0$;
4. If $d=4$, then up to projective equivalence, there are 5 possibilities for C, having respectively 1, 2 or 3 cusps. For example,

$$3A_2 : x^2y^2+y^2z^2+x^2z^2-2xyz(x+y+z)=0$$

$$A_2A_4 : z^4-xz^3-2xyz^2+x^2y^2=0, \quad A_6 : y^4-2xy^2z+yz^3+x^2z^2=0.$$

5. If $d=5$, then up to projective equivalence, there are 11 possibilities for C, having respectively 1, 2, 3 or 4 cusps. For $d=5$, any r.c.c. with 4 cusps is projectively equivalent to

$$3A_2A_6 : 16x^4y+128x^2y^2z-4x^3z^2+256y^3z^2-144xyz^3+27z^5=0.$$

Classification results I

A r.c.c. C is unicuspidal if it has a unique singularity. Here is the (topological) classification of such curves having a unique Puiseux pair.

Unique Puiseux pair (a, b) means same topology as the cusp $u^a + v^b = 0$, in particular $g.c.d.(a, b) = 1$. However, the analytic type, e.g. expressed by the Tjurina number

$$\tau(g) = \dim \mathcal{O}_2/(g_u, g_v, g) \leq \mu(g) = \dim \mathcal{O}_2/(g_u, g_v) = (a-1)(b-1)$$

can vary.

Theorem (J. Fernandez de Bobadilla, I. Luengo, A. Melle-Hernandez, A. Némethi, 2004)

Let a_i be the Fibonacci numbers with $a_0 = 0$, $a_1 = 1$, $a_{j+2} = a_{j+1} + a_j$. A Puiseux pair (a, b) can be realized by a unicuspidal rational curve of degree $d \geq 3$ if and only if the triple (a, b, d) occurs in the following list.

1. $(d-1, d, d)$;
2. $(d/2, 2d-1, d)$ *with* d *even;*
3. $(a_{j-2}^2, a_j^2, a_{j-2}a_j)$ *with* $j \geq 5$ *odd;*
4. (a_{j-2}, a_{j+2}, a_j) *with* $j \geq 5$ *odd;*
5. $(3, 22, 8)$ *and* $(6, 43, 16)$.

Classification results II

We say a r.c.c. C has type (d, m) if d is the degree of C and m is the highest multiplicity of its singularities. The classification of curves of type $(d, d-1)$ is rather easy, and the curves of type $(d, d-2)$ have been classified by Flenner-Zaidenberg (1996), Sakai-Tono (2000). The largest cusp can have the first Puiseux pair either $(d-2, d)$ (case (i) in the next result) or $(d-2, d-1)$ (case (ii)).

Here is a sample of this classification. Let C be a r.c.c. of type $(d, d-2)$ having two cusps, let's say q_1 of multiplicity $d-2$ and q_2 of multiplicity $\leq d-2$. Then the following cases are possible.

Proposition

(i) The germ (C, q_2) is a singularity of type A_{2d-4}.Then for each $d \geq 4$, C is unique up to projective equivalence.
(ii) The germ (C, q_2) is a singularity of type A_{d-1}, with $d = 2k+1 \geq 5$ odd. Up to projective equivalence the equation of C can be written as

$$C : f = (y^{k-1}z + \sum_{i=2,k} a_i x^i y^{k-i})^2 y - x^{2k+1} = 0.$$

(iii) The germ (C_d, q_2) is a singularity of type A_{2j}, with d even and $1 \leq j \leq (d-2)/2$. Up to projective equivalence the equation of C_d can be written as

$$C : f = (y^{k+j}z + \sum_{i=2,k+j+1} a_i x^i y^{k+j+1-i})^2 - x^{2j+1} y^{2k+1} = 0,$$

where $a_{k+j+1} \neq 0$, $d = 2k+2j+2 \geq 6$, $k \geq 0$, $j \geq 1$.

Conclusions

1. Classification of r.c.c. is very difficult and complex, maybe impossible.

2. However, the r.c.c. are very interesting objects of study. The Coolidge-Nagata conjecture (any r.c.c. can be transformed into a line using some birational morphisms of $\mathbb{P}^2$) was finally proved in 2015 by Mariusz Koras and Karol Palka.

And there are a number of interesting open questions, as the following one.

Question

What is the maximal number N of cusps a r.c.c. can have?

Known examples suggest $N = 4$, K. Tono has shown $N \leq 8$ (2005), K. Palka improved this bound to $N \leq 6$ (2014).

Free divisors in $\mathbb{P}^2$

Let J_f be the Jacobian ideal of f, i.e. the homogeneous ideal of S spanned by the partial derivatives f_x, f_y, f_z of f and let $M(f) = S/J_f$ be the corresponding graded ring, called the Jacobian (or Milnor) algebra of f.

Let I_f denote the saturation of the ideal J_f with respect to the maximal ideal $\mathbf{m} = (x, y, z)$ in S, i.e.

$$I_f = \{h \in S : \mathbf{m}^k h \subset J_f \text{ for some integer } k > 0\}.$$

Freeness in the local analytic setting was introduced by K. Saito (1980). Here we consider the graded version of this notion.

Definition

The curve $C : f = 0$ is a free divisor if the following two equivalent conditions hold.

1. $N(f) := I_f/J_f = H^0_{\mathbf{m}}(M(f)) = 0.$
2. The minimal resolution of the Milnor algebra $M(f)$ has the following (short) form

$$0 \to S(-d_1 - d + 1) \oplus S(-d_2 - d + 1) \to S^3(-d + 1) \to S.$$

When C is a free divisor, the integers $d_1 \leq d_2$ are called the exponents of C. They satisfy the relations

$$d_1 + d_2 = d - 1 \text{ and } d_1 d_2 = (d-1)^2 - \tau(C),$$

where $\tau(C)$ is the total Tjurina number of C.

First properties of free divisors

Let $m(f)_k = \dim M(f)_k$ and $n(f)_k = \dim N(f)_k$. It is known that $N(f)$ is self-dual, in particular $n(f)_k = n(f)_{T-k}$ for any k, where $T = 3(d-2)$. One can also show that

$$(\star) \qquad n(f)_k = m(f)_k + m(f)_{T-k} - m(f_s)_k - \tau(C),$$

where f_s is a homogeneous polynomial of degree d with $C_s : f_s = 0$ smooth, e.g. $f_s = x^d + y^d + z^d$.

Corollary

The freeness of $C : f = 0$ is determined by the sequence $m(f)_k$. Conversely, if $C : f = 0$ is free, the sequence $m(f)_k$ is determined by the exponents d_1 and d_2.

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Irreducible free divisors

The study of free line arrangements in $\mathbb{P}^2$ is a classical subject, see the papers by H. Terao (1981), M. Yoshinaga (2014), J. Vallès (2015). The central open question: is the freeness of a line arrangement $\mathcal{A}$ (or, in general, of a hyperplane arrangement) determined by the combinatorics, i.e. by the intersection lattice $L(\mathcal{A})$?

In this talk we concetrate however on irreducible free divisors in $\mathbb{P}^2$. Only few examples of irreducible free divisors were known until recently, see A. Simis and S. Tohăneanu (2011) and R. Nanduri (2013).

Example

The example given by A. Simis and S. Tohăneanu

$$C_d : f_d = y^{d-1}z + x^d + ax^2y^{d-2} + bxy^{d-1} + cy^d = 0, \quad a \neq 0$$

is a family of free rational cuspidal curves for $d \geq 5$.

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This example has the following properties:
(i) the rationality of C_d is obvious, since $z = R(x, y)$.
(ii) $\tau(C_d) < \mu(C_d)$ for $d > 5$.
We can show following.

Theorem

Any rational cuspidal curve C which is free of degree $d \geq 5$ satisfies

$$d_1 \geq 2 \text{ and } \mu(C) - \tau(C) = (d_1 - 1)(d_2 - 1) - 1.$$

In particular $\mu(C) > \tau(C)$ unless $d = 5$ and $d_1 = d_2 = 2$. Moreover, one has

$$\frac{3}{4}(d-1)^2 \leq \tau(C) \leq d^2 - 4d + 7.$$

Theorem (First experimental meta theorem)

About half of the rational cuspidal curves which occur in the various classification lists are free divisors.

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Nearly free divisors: definition and first properties

What about the other half?

Definition

The curve $C : f = 0$ is a nearly free divisor if $N(f) \neq 0$ and $n(f)_k \leq 1$ for any k.

Theorem

If $C : f = 0$ is nearly free, then the Milnor algebra $M(f)$ has a minimal resolution of the form

$$0 \to S(-d-d_2) \to S(-d-d_1+1) \oplus S^2(-d-d_2+1) \to S^3(-d+1) \to S$$

where the exponents $d_1 \leq d_2$ satisfy $d_1 + d_2 = d$ and $\tau(C) = (d-1)^2 - d_1(d_2-1) - 1$. In particular, there are 3 syzygies R_1, R_2, R_3 of degrees d_1, d_2, d_2 satisfying $hR_1 + \ell_2 R_2 + \ell_3 R_3 = 0$, with degree of h, ℓ_2, ℓ_3 equal to $d_2 - d_1 + 1, 1, 1$ and ℓ_2, ℓ_3 linearly independant.

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Theorem (Second experimental meta theorem)

All the rational cuspidal curves which occur in the various classification lists are either free or nearly divisors.

Example

The curve $C_d : f_d = x^d + y^{d-1}z = 0$ is nearly free for $d \geq 2$ with exponents $(d_1, d_2) = (1, d-1)$. In addition $n(f)_k = 1$ for $d-2 \leq k \leq 2d-4$ and $n(f)_k = 0$ otherwise.

Conjecture (First main conjecture)

Any rational cuspidal curve is either free or nearly free.

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Theorem (Main result)

Let $C : f = 0$ be a rational cuspidal curve of degree d. Assume that either

1. *d is even, or*
2. *d is odd and for any singularity x of C, the order of any eigenvalue λ_x of the local monodromy operator h_x is not d.*

Then C is either a free or a nearly free curve.

Corollary

Let $C : f = 0$ be a rational cuspidal curve of degree d such that

1. *either $d = p^k$ is a prime power, or*
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Example (rational cuspidal quintics)

Let $C : f = 0$ be a rational cuspidal curve of degree 5. Then the singularities of C are of the following types.

1. one cusp A_{12}, e.g. $x^4y + z^5 = 0$ (nearly free) or Simis-Tohăneanu example C_5 (free).
2. two cusps: E_8+A_4, E_6+A_6 or A_8+A_4.
3. three cusps: $E_6+A_4+A_2$ or $3A_4$.
4. four cusps: A_6+3A_2.

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Example (unicuspidal r.c.c.)

A unicuspidal rational curve with a unique Puiseux pair not of the type $(a, b, d) = (a_{j-2}^2, a_j^2, a_{j-2}a_j)$ with d odd is either free or nearly free.

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Conjecture (Second main conjecture)

An irreducible plane curve which either free or nearly free is rational.

Remark

Non-linear free arrangements can have non-rational irreducible components. Here is one example due to Jean Vallès.

$$xyz(x^3+y^3+z^3)[(x^3+y^3+z^3)^3-27x^3y^3z^3]=0.$$

Let $\nu : \tilde{C} \to C$ be the normalization of the irreducible curve C. Assume that the singularities of C are the points p_i, $i=1,...,q$ and the germ (C,p_i) has r_i branches.

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Theorem

(i) If C is an irreducible free curve of degree d, then $d \geq 5$ and

$$2g(\tilde{C}) + \sum_{i=1,q} (r_i - 1) \leq \frac{(d-1)(d-5)}{4}.$$

In particular, if $d \leq 6$, then C is rational and $r_i \leq 2$ for all i, with equality for at most one i.
(ii) If C is an irreducible nearly free curve of degree d such that $d_1 = 1$ (resp. $d \leq 5$), then C is rational cuspidal (resp. rational and $r_i \leq 2$ for all i, with equality for at most one i.).

There are examples of free (resp. nearly free) rational curves C having one singularity p_i with $p_i = 2$ (resp. $p_i = 3$).

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Relation to the monodromy

The preprint
U. Walther, The Jacobian module, the Milnor fiber, and the *D*-module generated by f^s, arXiv:1504.07164
implies the following result.

We have an injection

$$N(f)_{2d-2-j} \to H^2(F, \mathbb{C})_\lambda,$$

with $j = 1, 2, ..., d$, where $F : f(x, y, z) - 1 = 0$ is the Milnor fiber associated to C and the subscript λ indicates the eigenspace of the monodromy action corresponding to the eigenvalue $\lambda = \exp(2\pi i(d + 1 - j)/d)$.

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$\lambda = \exp(2\pi i(d + 1 - j)/d)$.

Proof in the case d even

Suppose now that d is even, say $d = 2d_1$. It is enough to show that $\dim H^2(F, \mathbb{C})_\lambda = 1$ for $\lambda = -1$ which corresponds to $j = d_1 + 1$.

Note first that -1 is not an eigenvalue for the local monodromy h_x of any cusp (C, x), e.g. using Lê's formula for the characteristic polynomial of the monodromy operator h_x in terms of the Puiseux pairs. A better proof suggested by David Massey: knot theory implies that $h_x - Id$ is a presentation matrix of the trivial $\mathbb{Z}$-module $H_1(S^3) = 0$, hence must be an isomorphism $\mathbb{Z}^{\mu_x} \to \mathbb{Z}^{\mu_x}$.

The fact that $\lambda = -1$ is not an eigenvalue for the local monodromy h_x of any cusp (C, x) implies that $H^1(F)_\lambda = 0$.
Since $E(U) = E(\mathbb{P}^2) - E(C) = 1$, it follows that

$$\dim H^2(F, \mathbb{C})_\lambda - \dim H^1(F, \mathbb{C})_\lambda + \dim H^0(F, \mathbb{C})_\lambda = 1.$$

Since clearly $H^0(F)_\lambda = 0$, the result is proved.

A Lefchetz type property for $N(f)$

The preprint
A. Dimca, D. Popescu, Hilbert series and Lefschetz properties of dimension one almost complete intersections, arXiv:1403.5921.
implies the following result, used twice in the proofs (case d even of the main theorem and the form of the minimal resolution of a nearly free divisor). Idea of proof: splitting of the syzygy bundle restricted to lines as in H. Brenner, A. Kaid (2007).

Theorem

If $C : f = 0$ is a degree d reduced plane curve, then there exists a Lefschetz element for $N(f)$. More precisely, for a generic linear form $\ell \in S_1$, the multiplication by ℓ induces injective morphisms $N(f)_i \to N(f)_{i+1}$ for integers $i < T/2$ and surjective morphisms $N(f)_i \to N(f)_{i+1}$ for integers $i \geq i_0 = [T/2]$. In particular one has

$$0 \leq n(f)_0 \leq n(f)_1 \leq ... \leq n(f)_{[T/2]} \geq n(f)_{[T/2]+1} \geq ... \geq n(f)_T \geq 0.$$

A Lefchetz type property for $N(f)$

The preprint

A. Dimca, D. Popescu, Hilbert series and Lefschetz properties of dimension one almost complete intersections, arXiv:1403.5921.

implies the following result, used twice in the proofs (case d even of the main theorem and the form of the minimal resolution of a nearly free divisor). Idea of proof: splitting of the syzygy bundle restricted to lines as in H. Brenner, A. Kaid (2007).

Theorem

If $C : f = 0$ is a degree d reduced plane curve, then there exists a Lefschetz element for $N(f)$. More precisely, for a generic linear form $\ell \in S_1$, the multiplication by ℓ induces injective morphisms $N(f)_i \to N(f)_{i+1}$ for integers $i < T/2$ and surjective morphisms $N(f)_i \to N(f)_{i+1}$ for integers $i \geq i_0 = [T/2]$. In particular one has

$$0 \leq n(f)_0 \leq n(f)_1 \leq ... \leq n(f)_{[T/2]} \geq n(f)_{[T/2]+1} \geq ... \geq n(f)_T \geq 0.$$